
Real Topological String Theory

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Abstract

The thesis comprises work done at SISSA (Trieste) and UAM (Madrid) under supervision of A. Uranga during academic years 2013-2016 and published in the following works:

- in the first one [79] we describe the type IIA physical realization of the unoriented topological string introduced by Walcher, describe its M-theory lift, and show that it allows to compute the open and unoriented topological amplitude in terms of one-loop diagram of BPS M2-brane states. This confirms and allows to generalize the conjectured BPS integer expansion of the topological amplitude. The M-theory lift of the orientifold is freely acting on the M-theory circle, so that integer multiplicities are a weighted version of the (equivariant subsector of the) original closed oriented Gopakumar-Vafa invariants. The M-theory lift also provides new perspective on the topological tadpole cancellation conditions. We finally comment on the M-theory version of other unoriented topological strings, and clarify certain misidentifications in earlier discussions in the literature.
- in the second [45] we consider the real topological string on certain non-compact toric Calabi-Yau three-folds \mathbb{X} , in its physical realization describing an orientifold of type IIA on \mathbb{X} with an O4-plane and a single D4-brane stuck on top. The orientifold can be regarded as a new kind of surface operator on the gauge theory with 8 supercharges arising from the singular geometry. We use the M-theory lift of this system to compute the real Gopakumar-Vafa invariants (describing wrapped M2-brane BPS states) for diverse geometries. We show that the real topological string amplitudes pick up certain signs across flop transitions, in a well-defined pattern consistent with continuity of the real BPS invariants. We further give some preliminary proposals of an intrinsically gauge theoretical description of the effect of the surface operator in the gauge theory partition function.

A Spanish translation follows.

Resumen

Esta memoria de tesis incluye trabajos hechos en SISSA (Trieste) y la UAM (Madrid) bajo la supervisión de Á. Uranga durante los años académicos 2013-2016 y publicados en los siguientes artículos:

- en el primero [79] se describe la realización física en teoría de cuerdas tipo IIA de la cuerda topológica no-orientada introducida por Walcher, su *uplift* a teoría M, y se muestra que se pueden calcular las amplitudes abiertas y no-orientadas en términos de diagramas a un *loop* de estados BPS constituidos por M2-branas. Esto confirma y permite generalizar la expansión entera BPS de la cuerda topológica, que fue conjeturada anteriormente. El *uplift* a teoría M del orientifold actúa libremente sobre el círculo de teoría M, por lo que las multiplicidades enteras son una versión ponderada de (sector equivariante de) los invariantes de Gopakumar y Vafa, que originalmente eran cerrados y orientados. El *uplift* a teoría M también nos da una nueva perspectiva sobre la condición de cancelación de *tadpoles*. Al final, comentamos otras versiones en teoría M de otras cuerdas topológicas no-orientadas, y aclaramos algunas imprecisiones en la literatura.
- en el segundo [45] consideramos la cuerda topológica real en espacios Calabi-Yau tóricos no-compactos \mathbb{X} , en su realización física descrita por medio de un orientifold en la teoría de cuerdas de tipo IIA en \mathbb{X} con un O4-plano y una D4-brana fija sobre el mismo. El orientifold se puede considerar como un nuevo tipo de operador de superficie en la teoría *gauge* con 8 *supercargas*, que provienen de la geometría singular. Usamos el *uplift* a teoría M de este sistema para calcular los invariantes reales de Gopakumar y Vafa (que describen estados BPS formados por M2-branas enrolladas) en diferentes geometrías. Demostramos que las amplitudes topológicas reales toman unos signos cuando se hacen transiciones de *flop*, en una manera bien definida y consistente con la continuidad de los invariantes BPS. También presentamos unas propuestas preliminares para la descripción intrínseca en la teoría *gauge* del efecto del operador de superficie en la función de partición de dicha teoría.

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Dedicated to my Family

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Chapter 1

Introduction

1.1. General setup

The topic of the thesis lies at the interface between theoretical high energy physics and geometry. Broadly speaking, we focus on topological string theory and its connections with supersymmetric gauge theories in four dimensions, in particular the counting of BPS states.

Recently this subfield of theoretical physics has seen many breakthroughs, both on the string side where topological vertex techniques have been developed to compute all-genera partition functions essentially via Feynman diagram tools, and on the susy side, where mathematical-solid arguments have been proposed by N. Nekrasov together with an explicit form (in terms of sums of Young diagrams) for the partition function of some twisted susy gauge theories: the development of such localization techniques allowed many exact results to be obtained, e.g. AGT correspondence [6, 93] relating Nekrasov instanton partition function on \mathbb{R}^4 with chiral blocks in Virasoro algebra.

There's an interesting duality, where one can 'engineer'¹ some gauge theory starting from *local* toric Calabi-Yau three-fold: non-compactness is necessary in order to decouple gravitational interactions that in large volume limit are suppressed due to the CY volume appearing in Planck mass coefficient of Einstein-Hilbert action after dimensional reduction; interesting physics comes from local singularity structure, namely there will some interacting cycles in the CY, which typically intersect with ADE pairing and support BPS branes.

One can then apply vertex techniques to show that topological string on the given CY gives precisely Nekrasov's result for the engineered gauge group: this stems from the fact that both theories are computing some Witten index taking contributions from BPS states, which are essentially the same for the two sides.

This circle of ideas is less understood for the case when unoriented and open contributions are present, namely when one has orientifold planes and D-branes wrapping them in the string construction, and is the topic of the present thesis.

1.2. Geometric engineering

We focus on $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions: their low-energy effective prepotential is holomorphic function of the Coulomb moduli a_i and masses of matter fields m_i , and receives contributions from space-time instantons. This class of theories can be engineered using type IIA superstring compactified on local Calabi-Yau 3-fold \mathbb{X} .

To obtain interesting non-abelian gauge groups, we include non-perturbative effects. One way [46, 60] is to consider D2-branes wrapped on compact 2-cycles F_i : they are electrically charged under the RR abelian gauge fields and have masses proportional to

¹Geometry of the compactification manifold determines the effective gauge theory in the field theory limit.

the volume t_{f_i} of the wrapped 2-cycles, so we need $t_{f_i} \rightarrow 0$ to have massless charged fields in the vector multiplet.

Regard \mathbb{X} as dimensional reduction of a K3 compactification near an ADE singularity. The gauge group originates in six dimensions from the compact homology of the singularity, with the intersection numbers of F_i equal to the Cartan matrix, while further dimensional reduction on $B = \mathbb{CP}^1$ leads to $\mathcal{N} = 2$ gauge theory in four dimensions. The bare gauge coupling and Kähler class of B are related by

$$t_b \sim 1/g_{\text{YM}}^2, \quad (1.2.1)$$

which follows from dimensional reducing a form from six to four dimensions. In order to decouple gravity and stringy effects it is sufficient to send the coupling constant to zero, since it pushes the string scale to infinity: this means that we take the limit

$$t_b \rightarrow \infty. \quad (1.2.2)$$

In order to satisfy the Calabi-Yau condition, the compactification space cannot be a direct product, but rather must be fibration

$$\begin{array}{ccc} F & \rightarrow & \mathbb{X} \\ & & \downarrow \\ & & B \end{array} \quad (1.2.3)$$

where the fiber geometry F (with the ADE singularity) determines the gauge group while the base geometry B the effective 4d gauge coupling. The fibration structure also allows to include matter via local enhancement of the fiber singularity.

The limits $t_b \rightarrow \infty$ and $t_{f_i} \rightarrow 0$ are not independent. Consider the example of pure $\text{SU}(2)$: to obtain the two charged gauge bosons it is sufficient to fiber a \mathbb{CP}_f^1 over the base \mathbb{CP}_b^1 .² In the weak coupling regime the running of the gauge coupling is given by

$$\frac{1}{g_{\text{YM}}^2} \sim \log \frac{m_W}{\Lambda}, \quad (1.2.4)$$

where m_W denotes the mass of the W-bosons and Λ the quantum scale. With the above identifications, we have to take the limit in a way such that $t_b \sim \log t_f$ holds. This example can also be used to understand eq. (1.2.2) from another perspective: from wrapping D2-branes over B we get W-bosons as well: these states are the light degrees of freedom of a different $\text{SU}(2)_b$ theory, which however appears at strong coupling of the six dimensional $\text{SU}(2)_f$ theory from \mathbb{CP}_f^1 , since its Coulomb parameter is given by t_b in eq. (1.2.1). In general this theory will decouple in the $t_b \rightarrow \infty$ limit, which we should take in going from string theory to the field theory limit with gauge group $\text{SU}(2)_f$.

²The different ways this can be done are labeled by an integer and the corresponding geometries correspond to Hirzebruch surfaces. The Calabi-Yau 3-fold is the total space of the anti-canonical bundle over this complex surface. All Hirzebruch surfaces give rise to pure $\text{SU}(2)$ in four dimensions, with different Chern-Simons level if seen from the 5d perspective.

The useful property of the type IIA string construction is that space-time instanton corrections are mapped to world-sheet instanton corrections: the Euclidean string world-sheet is wrapped around the 2-cycles of the geometry with worldsheet instanton action $S \sim d_b t_b + d_f t_f$, where d_b and d_f refer to wrapping numbers. This means that we do not need to consider the full type IIA string theory to investigate the gauge theory. Rather, the topological sector is sufficient, i.e. the topological string amplitudes that capture world-sheet instanton corrections.

Since the geometric engineering limit involves $t_{f_i} \rightarrow 0$, the compactification geometry is singular, and we are not expanding the string amplitudes around the large volume point in moduli space. Hence, if we compute the topological string amplitudes using A-model techniques, such as localization or the topological vertex, which are valid at large volume, these amplitudes have to be analytically continued before we can take the limit.

There's an equivalent description [43, 58, 92] in IIA string theory: at the origin of Coulomb branch, an $\mathcal{N} = 2$ gauge theory with gauge group $U(N)$ can be realized on a stack of D4-branes stretched between parallel NS5-branes, as in fig. 1.1. This produces an effective 4 dimensional world-volume for the D4 branes and $\mathcal{N} = 2$ supersymmetry is preserved in 4 dimensions. The D4-branes sit at the origin of the $x_4 + ix_5$ coordinate,

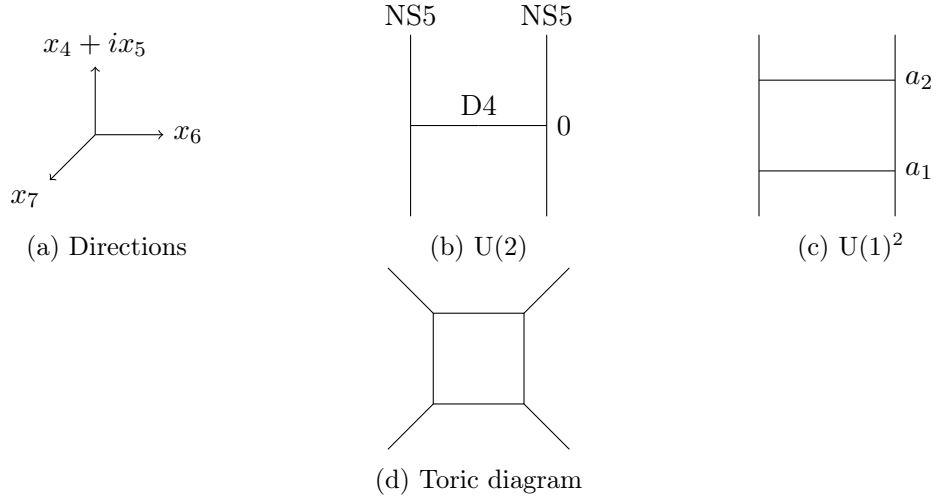


Figure 1.1: Equivalence for $N = 2$.

and the distance between NS5-branes in the x_6 direction (complexified to $x_6 + ix_{10}$ in an M-theory lift), measured at various positions in $x_4 + ix_5$, is the running gauge coupling. At a generic point on the Coulomb branch, the D4-branes move apart, with $x_4 + ix_5$ positions corresponding to the eigenvalues of the adjoint scalar. The end points of the D4 branes on the NS5 branes are singular. However an approximate description is that the D4 branes exert a force on the NS5 branes causing them to bend. The resulting theory is then engineered in a toric geometry [68], where the skeletons of toric space

are identified with the brane configurations. Differences in Coulomb eigenvalues become fiber Kähler parameters and the scale Λ becomes a base Kähler parameter.

1.3. Topological strings

We now briefly review (standard, i.e. closed oriented) topological string theory [17]: the prepotential for the three-point function on the sphere can be written in terms of classical intersection numbers and genus-zero Gromov-Witten invariants,

$$F_0 = \frac{1}{3!} c_{abc} t^a t^b t^c + \sum_{0 \neq \beta \in H_2(\mathbb{X}; \mathbb{Z})} N_\beta^0 e^{-\int_\beta \omega} \quad (1.3.1)$$

and it is not a monodromy invariant quantity, so the subleading classical terms are always given only up to monodromy transformations; namely, F_0 is ambiguous up to $P_2(t) = -\frac{\chi(\mathbb{X})}{2} \zeta(3) - \frac{\pi^2}{6} \int_{\mathbb{X}} c_2(\mathbb{X}) \wedge k + A_{ij} t^i t^j$; at genus one we have

$$F_1 = -\frac{1}{24} t^a \int_{\mathbb{X}} c_2(\mathbb{X}) \wedge \omega_a + \sum_{0 \neq \beta \in H_2(\mathbb{X}; \mathbb{Z})} N_\beta^1 e^{-\int_\beta \omega}, \quad (1.3.2)$$

and for higher genera

$$F_{g \geq 2} = (-1)^g \frac{\chi(\mathbb{X})}{2} \int_{\mathcal{M}_g} \lambda_{g-1}^3 + \sum_{0 \neq \beta \in H_2(\mathbb{X}; \mathbb{Z})} N_\beta^g e^{-\int_\beta \omega}, \quad (1.3.3)$$

where

$$\int_{\mathcal{M}_g} \lambda_{g-1}^3 = -\frac{B_{2g} B_{2g-2}}{2g(2g-2)(2g-2)!}. \quad (1.3.4)$$

Although several techniques are available,³ the quantities $N_\beta^g \in \mathbb{Q}$ are in general hard to compute; for local toric CY threefolds \mathbb{X} , one can use so-called topological vertex: set $q = e^{ig_s}$ and $n_{j_L}^\beta = \sum_{j_R} (-1)^{2j_R} (2j_R + 1) N_\beta^{j_L, j_R}$, where the numbers $N_\beta^{j_L, j_R} \in \mathbb{Z}$ are related to refined counting of BPS states; if we define the topological string partition function $Z = \exp \sum_{g=0}^\infty g_s^{2g-2} F_g$, then we get

$$Z = \frac{\exp \left(\frac{1}{3!} c_{abc} t^a t^b t^c - \frac{1}{24} t^a \int_{\mathbb{X}} c_2(\mathbb{X}) \wedge \omega_a \right)}{\exp \left(-\frac{\zeta(3)}{g_s^2} + \sum_{g=2}^\infty g_s^{2g-2} (-1)^g \int_{\mathcal{M}_g} \lambda_{g-1}^3 \right)^{-\frac{\chi(\mathbb{X})}{2}}} \prod_{\substack{\beta, j_L \\ k_L = -j_L \\ m=1}} \left(1 - q^{2k_L + m} e^{-\int_\beta \omega} \right)^{m(-1)^{2j_L} n_\beta^{j_L}} \quad (1.3.5)$$

³For example, localization in the A-model, holomorphic anomaly in the B-model, constraints from modular invariance, and asymptotic properties of generating function series.

1.4. The project

The PhD project is related to the unoriented A-model topological string that J. Walcher has worked on [91]. The basic idea is that one would like to include in the perturbative sum over genera also surfaces with boundaries and crosscaps, and count (appropriately defined) holomorphic maps from these surfaces to a Calabi-Yau three-fold target.

At the computational level, one can either employ localization techniques in the A-model setup (essentially a modified version of results by Kontsevich) or apply mirror symmetry and integrate the B-model holomorphic anomaly equations: one is forced by mathematical consistency (related to the appearance of otherwise ill-defined quantities, and to the existence of real codimension one boundaries in moduli space; these issues have also been studied by Georgieva and Zinger [32, 33]) to combine the rational numbers so obtained, and interestingly it's possible to do that in a consistent way and get integer multiplicities, which suggest a possible BPS interpretation, in the spirit of what Gopakumar-Vafa, and Ooguri-Vafa did for closed oriented and open oriented strings respectively.

To define the model properly, one must impose some “topological tadpole cancellation conditions” that correspond to having one boundary, i.e. one D-brane, as seen in the covering. The unoriented topological A-model has a physical theory corresponding to type IIA with an O4-plane, wrapped on a lag 3-cycle of the CY. The “tadpole condition” requires to introduce one D4-brane on the same 3-cycle, and the question is what is special about having one (rather than more, or less) D4.

Our earlier result [79] shows that the configuration of one O4 and one D4 has an interesting lift to M-theory, specifically a purely geometrical background corresponding to an orbifold by the antiholomorphic involution on the CY times a half-shift on the M-theory circle. This allows for a physical definition of the unoriented Gopakumar-Vafa invariants that Walcher (together with Krefl as well) computes by localization and holomorphic anomaly matching, and it explains the cancellation mechanism from a physical viewpoint, namely by canceling contributions from curves that are equal except for trading a crosscap for a disk by means of a relative sign produced by the C_3 form in M-theory surrounding a crosscap, and suggesting a way to glue along real codimension one boundaries in moduli space.

One can also study the above system from the gauge theory viewpoint, and ask whether there's some relation between the real topological string and some modified version of Nekrasov instanton partition function. This is expected on general grounds (duality between topological strings on CY3's and $\mathcal{N} = 2$ gauge theories in 4d/5d) basing on geometric engineering arguments and decoupling of gravity.

What we discovered [45] is that one can start from $\mathcal{N} = 2$ gauge theory in 4d and perform a shift in one of the Ω -background equivariant parameters, corresponding to the directions transverse to the O4-plane in spacetime. This matches almost exactly (see the subtlety below) the real topological string computations for local toric CY3's, which can be done with the help of a modified version of the topological vertex taking into

account the quotient by toric involutions [65], and it suggests a possible gauge theory interpretation for the real topological string.

A natural question one could ask, from the physical superstring viewpoint, is whether the real topological amplitude computes any effective term/correction for superstring in selfdual graviphoton background, similarly to what has been done for the closed oriented case [9]. The answer seems to be positive, and we are able to understand the physical meaning of open and unoriented topological amplitudes: one can check that the relevant RR operators survive orientifold projection, and then the real topological string computation is mapped via a covering trick to a computation on a closed oriented Riemann surface, but restricted to left-movers only. One can also study the heterotic dual of this, where the \sinh^{-1} power that was previously found in the 2d Schwinger computation can be recovered.

1.5. Organization

The rest of the thesis is organized as follows.

Chapter 2 deals with the M-theory interpretation of the real topological string. In section 2.2 we review the Gopakumar-Vafa reformulation of the closed oriented topological A-model. In section 2.3 we review the properties of the real topological string. In section 2.4 we present the physical IIA theory corresponding to this topological model and construct its M-theory lift (section 2.4.1), compute the partition function in terms of M-theory BPS invariants (section 2.4.2), and describe the M-theory explanation of the tadpole cancellation conditions (section 2.4.3). In section 2.5 we describe related systems, by the inclusion of additional brane pairs (section 2.5.1), or by using other orientifold plane structures (section 2.5.2); in this respect, we clarify certain misidentifications of the M-theory lifts in the earlier literature on unoriented topological models. Section 2.A reviews the basics of the real topological string, while section 2.B discusses the physical couplings computed by the real topological string.

Chapter 3 is about the connection with gauge theory. In section 3.2 we review the M-theory/gauge theory correspondence in the oriented case: in section 3.2.1 we review the topological vertex computation of topological string partition functions on local CY three-folds, in section 3.2.2 we describe the computation of the gauge theoretic Nekrasov partition function via localization. In section 3.3 we review the computation of real topological string amplitudes: section 3.3.1 introduces some general considerations of unoriented theories, and section 3.3.2 describes the real topological string computation using the real topological vertex. Explicit examples are worked out in section 3.4, like the conifold ($U(1)$ gauge theory) in section 3.4.1, where we correct some typos in the previously known result, and the pure $SU(N)$ theories in section 3.4.2, where we also discuss their behavior under flop transitions. In section 3.5 we describe a twisted Nekrasov partition function, whose structure is motivated by the action of the orientifold, and compare it with the real topological string partition function. Section 3.A reviews aspects of the real topological string and the topological vertex formulation, section 3.B presents

some new enumerative checks of the BPS integrality, and section 3.C gathers some useful identities.

Finally, we offer our conclusions in chapter 4.

Chapter 2

M-theory interpretation

2.1. Introduction

Topological string theory is a fertile arena of interplay between physics and mathematics. A prominent example is the physics-motivated reformulation of the topological A-model on a threefold \mathbb{X}_6 in terms of integer multiplicities of BPS states in the 5d compactification of M-theory on \mathbb{X}_6 [36, 37], and the corresponding mathematical reformulation of the (in general fractional) Gromov-Witten invariants in terms of the *integer* Gopakumar-Vafa invariants (see also [51]).

A natural generalization is to consider A-models with different worldsheet topologies. In particular, there is a similar story for the open topological A-model, in which worldsheets are allowed to have boundaries mapped to a lagrangian 3-cycle in \mathbb{X}_6 , and which via lift to M-theory admits an open BPS invariant expansion [77]. There has also been substantial work to define unoriented topological A-models, for instance in terms of the so-called real topological strings [65–67, 91]. The latter was proposed to require a specific open string sector for consistency, and conjectured to admit a BPS-like expansion ansatz, although no physical derivation in terms of M-theory was provided. Conversely, although some unoriented topological models have been proposed directly from the M-theory picture [1, 4, 20, 21, 81], they do not correspond to this real topological string.

We fill this gap, construct the physical theory corresponding to the real topological string, and show that its M-theory lift reproduces the topological string partition function in terms of certain BPS invariants, which we define and show to be the equivariant subsector of the corresponding closed oriented Gopakumar-Vafa invariants. Along the way, the M-theory picture sheds new light into certain peculiar properties of the topological model, like the so-called tadpole cancellation condition, which requires combining open and unoriented worldsheets in order to produce well-defined amplitudes and integer invariants. Although [91] focused on the quintic and other simple examples (see also [65–67]), we keep the discussion general, using these examples only for illustration at concrete points.

2.2. Review of Gopakumar-Vafa expansion

We start with a brief review of the Gopakumar-Vafa interpretation of the closed oriented topological string in terms of BPS states in M-theory [36, 37].

The 4d compactification of type IIA on a CY threefold \mathbb{X}_6 provides a physical realization of the topological A-model on \mathbb{X}_6 , whose genus g partition function $F_g(t_i)$, which depends on the Kähler moduli t_i , computes the F-term

$$\int d^4x \int d^4\theta F_g(t_i) (\mathcal{W}^2)^g \rightarrow \int d^4x F_g(t_i) F_+^{2g-2} R_+^2 \quad (2.2.1)$$

(where the second expression applies for $g > 1$ only). Here we have used the $\mathcal{N} = 2$ Weyl multiplet, schematically $\mathcal{W} = F_+ + \theta^2 R_+ + \dots$, with F_+ , R_+ being the self-dual components of the graviphoton and curvature 2-form, respectively. These contributions

are summed up if we turn on a self-dual graviphoton background in the four non-compact dimensions

$$F_+ = \frac{\lambda}{2} dx^1 \wedge dx^2 + \frac{\lambda}{2} dx^3 \wedge dx^4. \quad (2.2.2)$$

The sum is given by the total A-model partition function, with coupling λ

$$\mathcal{F}(t_i) = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t_i). \quad (2.2.3)$$

There is an alternative way to compute this same quantity, by considering the lift of the IIA configuration to M-theory, as follows. We start with the 5d compactification of M-theory on \mathbb{X}_6 . There is a set of massive half BPS particle states, given by either the dimensional reduction of 11d graviton multiplets, or by M2-branes wrapped on holomorphic 2-cycles. These states are characterized by their quantum numbers under the 5d little group $SU(2)_L \times SU(2)_R$. Note that at the classical level, each such particle can have a classical moduli space, but at the quantum level there is only a discrete set of ground states, which provide the BPS particle states we are interested in. For instance, an 11d particle (such as the 11d graviton) has a classical moduli space given by \mathbb{X}_6 itself, but quantization leads to wave functions given by the cohomology of \mathbb{X}_6 , resulting in a net BPS multiplicity given by $\chi(\mathbb{X}_6)$.

In order to relate to type IIA, we compactify on an \mathbb{S}^1 . Corrections to the R^2 term will arise from one-loop diagrams in which the above BPS particles run, in the presence of the graviphoton field, which couples to their $SU(2)_L$ quantum numbers. In type IIA language this corresponds to integrating out massive D0- and D2-brane states (and their bound states). In the Schwinger proper time formalism we have

$$\begin{aligned} \mathcal{F} &= \int_{\epsilon}^{\infty} \frac{ds}{s} \text{tr}_{\mathcal{H}} \left[(-1)^F e^{-s(\Delta + m^2 + J \cdot F_+)} \right] \\ &= \int_{\epsilon}^{\infty} \frac{ds}{s} \sum_{k \in \mathbb{Z}} \frac{1}{4 \sinh^2\left(\frac{s\lambda}{2}\right)} \text{tr}_{\mathcal{H}} \left[(-1)^F e^{-2s\lambda J_3^L - sZ - 2\pi i s k} \right]. \end{aligned} \quad (2.2.4)$$

Here the \sinh^2 factor arises from the 4d kinematics, we have included a sum over KK momenta along the \mathbb{S}^1 , the trace is over the Hilbert space \mathcal{H} of 5d one-particle BPS states, with central charge Z , and $F = 2J_3^L + 2J_3^R$.

The Hilbert space \mathcal{H} of 5d one-particle BPS states from an M2-brane on a genus g holomorphic curve Σ_g (in general not the same as the genus of the worldsheet in the type IIA interpretation) in the homology class β is obtained by quantization of zero modes on its worldline. Quantization of the universal Goldstinos contributes to the state transforming as a (half) hypermultiplet, with $SU(2)_L$ representation $I_1 = \left(\frac{1}{2}\right) \oplus 2(0)$. There are in general additional zero modes, characterized in terms of the cohomology groups

$$\mathcal{H} = H^{\bullet}(\mathcal{M}_{g,\beta}) \otimes H^{\bullet}(\mathbb{T}^{2g}). \quad (2.2.5)$$

The first factor corresponds to zero modes from the deformation moduli space $\mathcal{M}_{g,\beta}$ of Σ_g in \mathbb{X}_6 , whose quantization determines the $SU(2)_R$ representation. The latter is decoupled

from the self-dual graviphoton background, so it only contributes as some extra overall multiplicity in the above trace.¹

The second factor corresponds to zero modes arising from flat connections on the type IIA D2-brane worldvolume gauge field on Σ_g . The \mathbb{T}^{2g} should be regarded as the Jacobian of Σ_g , $\text{Jac } \Sigma_g = \mathbb{T}^{2g}$. Quantization of these zero modes determines further contributions to the $\text{SU}(2)_L$ representation of the state as dictated by the $\text{SU}(2)$ Lefschetz decomposition of cohomology of \mathbb{T}^{2g} , i.e. with creation, annihilation and number operators

$$J_+ = k \wedge, \quad J_- = k \lrcorner, \quad J_3 = (\deg - n)/2. \quad (2.2.6)$$

Here k is the Kähler form of the torus, \lrcorner denotes contraction, the bidegree \deg is $p + q$ for a (p, q) -form and n is complex dimension.

The $\text{SU}(2)$ representation is of the form $I_g = I_1^{\otimes g}$, where $I_1 = (\frac{1}{2}) \oplus 2(0)$. For instance, for $g = 1$ we have a ground state 1 and operators dz and $d\bar{z}$, so that the cohomology of \mathbb{T}^2 splits as

$$\begin{pmatrix} 1 \\ k \end{pmatrix} \quad dz \pm d\bar{z} \quad (2.2.7)$$

where $k \sim dz \wedge d\bar{z}$. These form the representation I_1 . The argument generalizes straightforwardly to higher genera.

The contribution from a state in the $\text{SU}(2)_L$ representation I_g to the trace is given by $(-4)^g \sinh^{2g} \frac{s\lambda}{2}$, so we get

$$\mathcal{F} = - \int_{\epsilon}^{\infty} \frac{ds}{s} \sum_{g, \beta} \sum_{k \in \mathbb{Z}} \text{GV}_{g, \beta} \left(2i \sinh \frac{s\lambda}{2} \right)^{2g-2} e^{-s\beta \cdot t} e^{-2\pi i s k}, \quad (2.2.8)$$

where we write $Z = \beta \cdot t$. Also, $\text{GV}_{g, \beta}$ are integers describing the multiplicity of BPS states arising from M2-branes on a genus g curve in the class $\beta \in H_2(\mathbb{X}_6; \mathbb{Z})$, with the understanding that $\beta = 0$ corresponds to 11d graviton states. This multiplicity includes that arising from the $\text{SU}(2)_R$ representations, in the following sense. Describing the set of BPS states in terms of their $\text{SU}(2)_L \times \text{SU}(2)_R$ representations

$$\left[\left(\frac{1}{2}, 0 \right) \oplus 2(0, 0) \right] \otimes \sum_{j_L, j_R} N_{j_L, j_R}^{\beta} (j_L, j_R) \quad (2.2.9)$$

we have

$$\sum_{j_L, j_R} (-1)^{2j_R} (2j_R + 1) N_{j_L, j_R}^{\beta} [j_L] = \sum_g \text{GV}_{g, \beta} I_g. \quad (2.2.10)$$

Going back to eq. (2.2.8), we use Poisson resummation

$$\sum_{k \in \mathbb{Z}} e^{-2\pi i s k} = \sum_{m \in \mathbb{Z}} \delta(s - m) \quad (2.2.11)$$

¹This is no longer true if one considers refined topological strings as in [2, 4, 24, 40, 56], corresponding to a non self-dual background field of the form $F = \epsilon_1 dx^1 \wedge dx^2 - \epsilon_2 dx^3 \wedge dx^4$.

to get

$$\mathcal{F} = - \sum_{g,\beta} \sum_{m=1}^{\infty} \text{GV}_{g,\beta} \frac{1}{m} \left(2i \sinh \frac{m\lambda}{2} \right)^{2g-2} e^{-m\beta \cdot t}. \quad (2.2.12)$$

This is known as the GV or BPS expansion of the closed oriented topological string amplitude. There are similar expansions for open oriented topological string amplitudes (see section 2.5.1 for more details). The situation for unoriented topological string amplitudes is the main topic of this chapter.

2.3. Walcher's real topological string

A prominent example of unoriented topological string is Walcher's real topological string introduced in [91] (see also [65–67]). It includes both open and closed unoriented topological strings, subject to a mysterious ‘tadpole cancellation condition’ requiring the open string sector to be described by a single D-brane on top of the fixed locus of the orientifold action. We now review its basic features, the proposed tadpole cancellation condition, and the conjectured expansion in terms of integer multiplicities. For other details, see section 2.A. For simplicity, we take the case of a single Kähler modulus, although the generalization is straightforward. Also, examples in the literature [65–67, 91] have considered cases with $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$, for instance the quintic, or local \mathbb{CP}^2 . Since our description in M-theory is more general, and this condition will only play a role in section 2.4.3, we keep the description general here as well.

The A-model target space is a Calabi-Yau threefold \mathbb{X}_6 , equipped with an antiholomorphic involution σ , whose pointwise fixed set is a lagrangian 3-cycle denoted by L . The model is defined by considering maps of (possibly non-orientable) surfaces (possibly with boundaries) into \mathbb{X}_6 , with boundaries lying in L (so that the maps are topologically classified by the relative homology $d = f_*([\Sigma]) \in H_2(\mathbb{X}_6, L; \mathbb{Z})$), and with compatible orientifold actions on the target and worldsheet, as follows. We construct the non-orientable surface Σ as the quotient of the parent oriented Riemann surface $\widehat{\Sigma}$ by an antiholomorphic involution Ω of the worldsheet,² and demand equivariance of the holomorphic map f as in fig. 2.1.

$$\begin{array}{ccc} \widehat{\Sigma} & \xrightarrow{\Omega} & \widehat{\Sigma} \\ f \downarrow & \circlearrowleft & \downarrow f \\ \mathbb{X}_6 & \xrightarrow{\sigma} & \mathbb{X}_6 \end{array}$$

Figure 2.1: Equivariance means the diagram is commutative (we are improperly calling f both the equivariant map and its lift $\Sigma \rightarrow X/\sigma$.)

²This is equivalent to considering a dianalytic structure on the surface, which generalizes the notion of complex structure by allowing for antiholomorphic transition functions.

In the relation $\Sigma = \widehat{\Sigma}/\Omega$, the particular case in which Σ is itself closed oriented and $\widehat{\Sigma}$ has two connected components is not included.

The topological classification of possibly non-orientable surfaces Σ with boundaries, described as symmetric Riemann surfaces, written $(\widehat{\Sigma}, \Omega)$, generalizes the closed oriented case, through the following classic result:

Theorem 2.3.1. *Let $h = h(\widehat{\Sigma}, \Omega) \geq 0$ be the number of components of the fixed point set $\widehat{\Sigma}_\Omega$ of Ω in $\widehat{\Sigma}$ (i.e. the number of boundaries of Σ), and introduce the index of orientability $k = k(\widehat{\Sigma}, \Omega)$, given by $\left(2 - \# \text{ components of } \widehat{\Sigma} \setminus \widehat{\Sigma}_\Omega\right)$. Then the topological invariants h and k together with the genus \hat{g} of $\widehat{\Sigma}$ determine the topological type of $(\widehat{\Sigma}, \Omega)$ uniquely. For fixed genus \hat{g} , these invariants satisfy*

- (I) $k = 0$ or $k = 1$ (corresponding to oriented surfaces, or otherwise)
- (II) if $k = 0$, then $0 < h \leq \hat{g} + 1$ and $h \equiv \hat{g} + 1 \pmod{2}$
- (III) if $k = 1$ then $h \leq \hat{g}$.

Let us define the (negative of the) Euler characteristic of $\widehat{\Sigma}/\Omega$ by $\chi = \hat{g} - 1$. It is useful to separate the worldsheets into three classes, corresponding to having 0, 1 or 2 crosscaps (recall that two crosscaps are equivalent to a Klein handle, namely two holes glued together with an orientation reversal, which in the presence of a third crosscap can be turned into an ordinary handle). This leads to a split of the topological amplitudes into classes, namely: closed oriented surfaces (with amplitude denoted by $\mathcal{F}^{(g_\chi)}$, with $g_\chi = \frac{1}{2}\chi + 1$ the number of handles), oriented surfaces with h boundaries (with amplitude $\mathcal{F}^{(g,h)}$), non-orientable surfaces with an odd number of crosscaps (with amplitude $\mathcal{R}^{(g,h)}$) and non-orientable surfaces with an even number of crosscaps ($\mathcal{K}^{(g,h)}$). The Euler characteristic is given by $\chi = 2g - 2 + h + c$, with $c = 0, 1$.³

The basic tool used to compute these amplitudes is equivariant localization on the moduli space \mathcal{M} of stable maps, following ideas going back to [64] (see also [27], and [22, 34] for more recent developments on the formal side). Localization is with respect to a torus action which is compatible with the involution, and leads to a formulation in terms of the diagram techniques of [64].

Ref. [91] finds that, in the example of the quintic or local \mathbb{CP}^2 , in order to apply this machinery to unoriented and/or open worldsheets, some constraints, dubbed *tad-pole cancellation* conditions, have to be imposed: as we discuss in section 2.A, this is a cancellation between contributions from worldsheets with an unpaired crosscap and worldsheets with boundaries, with one boundary ending on L ‘with even degree’ (specifically, wrapping the generator of $H_1(L, \mathbb{Z}) = \mathbb{Z}_2$ an even number of times, hence begin topologically trivial). This results in a condition

$$d \equiv h \equiv \chi \pmod{2}, \quad (2.3.1)$$

³Note that we define g such that the negative Euler characteristic is $2g + h - 2$, $2g + h - 1$, and $2g + h - 2$ in the \mathcal{F} , \mathcal{R} and \mathcal{K} cases respectively, i.e. it also accounts for Klein handles.

where $d \in H_2(\mathbb{X}, L; \mathbb{Z}) = \mathbb{Z}$ is the relevant homology class. It implies that \mathcal{R} -type amplitudes do not contribute, $\mathcal{R}^{(g,h)} \equiv 0$.

Mathematically, this condition applies to real codimension one boundary strata in moduli space, in which a given worldsheet piece near L develops a node which can be smoothed to yield either a disk or a crosscap. The combined count of these homologically trivial disks and crosscaps leads to cancellation of potentially ill-defined pieces, and produces an invariant count.

Strong evidence for this consistency condition comes from the fact that the invariant numbers thus computed turn out to be all integers. This motivated the proposal of an ansatz reminiscent of a BPS expansion, as a sum over holomorphic *embeddings* (rather than maps) equivariant with respect to worldsheet parity $\Omega : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$.

If we write the total topological amplitude as

$$\mathcal{G}^{(\chi)} = \frac{1}{2} \left[\mathcal{F}^{(g_\chi)} + \sum \mathcal{F}^{(g,h)} + \sum \mathcal{K}^{(g,h)} \right], \quad (2.3.2)$$

the conjecture is

$$\sum_{\chi} i^{\chi} \lambda^{\chi} \left(\mathcal{G}^{(\chi)} - \frac{1}{2} \mathcal{F}^{(g_\chi)} \right) = \sum_{\substack{\chi \equiv d \pmod{2} \\ k \text{ odd}}} \widetilde{\text{GV}}_{\hat{g}, d} \frac{1}{k} \left(2 \sinh \frac{\lambda k}{2} \right)^{\chi} q^{kd/2}. \quad (2.3.3)$$

In more physical terms, the tadpole cancellation condition means that the background contains a single D-brane wrapped on L , as counted in the covering space. The interpretation in terms of a physical type IIA construction and its lift to M-theory will be discussed in the next section.

2.4. M-theory lift and BPS expansion

2.4.1. Tadpole cancellation, the O4/D4 system and M-theory

It is natural to look for a physical realization of the real topological string in terms of type IIA on the threefold \mathbb{X}_6 , quotiented by worldsheet parity times an involution acting antiholomorphically on \mathbb{X}_6 . In general, we consider involutions with a fixed point set along the lagrangian 3-cycle L , which therefore supports an orientifold plane. The total dimension of the orientifold plane depends on the orientifold action in the 4d spacetime, and can correspond to an O6-plane or an O4-plane. The choice of 4d action is not specified in the topological string, but can be guessed as follows.

We expect that the topological tadpole cancellation condition has some translation in the physical theory, as a special property occurring when precisely one D-brane (as counted in the covering space) is placed on top of the orientifold plane. Since the charge of a negatively charged O4-plane is -1 (in units of D4-brane charge in the covering), the configuration with a single D4-brane stuck on top of it is special, because it cancels the RR charge locally (on the other hand, the charge of an O6-plane is -4 , and no similarly special property occurs for a single stuck D6-brane).

The presence of a single D4-brane stuck on the O4-plane is not a consistency requirement of the type IIA theory configuration,⁴ but rather a condition that we will show leads to a particularly simple M-theory lift, and a simple extension of the Gopakumar-Vafa BPS expansion of topological amplitudes. This nicely dovetails the role played by tadpole cancellation in the real topological string to achieve the appearance of integer invariants.

The M-theory lift of O4-planes with and without D4-branes has been discussed in [35, 47]. In particular, a negatively charged O4-plane with no stuck D4-brane, spanning the directions 01234 in 10d Minkowski space M_{10} , lifts to M-theory on a \mathbb{Z}_2 orbifold $M_5 \times (\mathbb{R}^5/\mathbb{Z}_2) \times \mathbb{S}^1$ with generator $(x^5, \dots, x^9) \rightarrow (-x^5, \dots, -x^9)$, and which also flips the M-theory 3-form, $C_3 \rightarrow -C_3$. The latter action is required to be a symmetry of the M-theory Chern-Simons term, and matches the effect of the type IIA orientifold action on the NSNS 2-form B_2 . Hence, we will classify the M-theory action as ‘orientifold’ as well.

The construction generalizes to compactification on \mathbb{X}_6 , with the orientifold acting holomorphically on \mathbb{X}_6 . It produces M-theory on the quotient $(M_4 \times \mathbb{X}_6)/\mathbb{Z}_2 \times \mathbb{S}^1$, with the \mathbb{Z}_2 acting as the antiholomorphic involution σ on \mathbb{X}_6 and as $x^2, x^3 \rightarrow -x^2, -x^3$ on 4d Minkowski space. This system, and its generalization with additional D4-brane pairs (M5-branes in M-theory), is discussed in section 2.5.2. Here we simply note that the explicit breaking of the $SU(2)_L$ symmetry already in the 5d theory makes necessary to make certain assumptions on the structure of BPS multiplets in the theory, obscuring the derivation of the BPS expansion of the amplitude.

The M-theory lift of a negatively charged O4-plane with a stuck D4-brane is however much simpler, and in particular does not suffer from these difficulties. Because of the already mentioned local cancellation of the RR charge, the M-theory lift is a completely smooth space described by a freely acting quotient $M_5 \times (\mathbb{R}^5 \times \mathbb{S}^1)/\mathbb{Z}_2$, with generator acting as $(x^5, \dots, x^9) \rightarrow (-x^5, \dots, -x^9)$, as a half-period shift on \mathbb{S}^1 ($y \rightarrow y + \pi$ for periodicity $y \simeq y + 2\pi$), and flipping the 3-form C_3 (hence defining an M-theory orientifold).

Concerning the latter, it is important to point out that the negative charge of the O4-plane implies that there is a half-unit NSNS B_2 background on an \mathbb{RP}^2 surrounding the O4-plane; consequently, there is a non-trivial half-unit of 3-form background on the corresponding M-theory lift $(\mathbb{CP}^1 \times \mathbb{S}^1)/\mathbb{Z}_2$. This will play an important role in the M-theory interpretation of the disk/crosscap tadpole cancellation, see section 2.4.3.

The construction generalizes to compactification on \mathbb{X}_6 , with the orientifold acting holomorphically on \mathbb{X}_6 . It produces M-theory on the quotient $(M_4 \times \mathbb{X}_6 \times \mathbb{S}^1)/\mathbb{Z}_2$, with the \mathbb{Z}_2 acting as

$$\begin{aligned} \mathbb{X}_6 : \quad & x \mapsto \sigma(x), & \mathbb{S}^1 : \quad & y \mapsto y + \pi, \\ \text{Minkowski} : \quad & x^0, x^1 \mapsto x^0, x^1, & & x^2, x^3 \mapsto -x^2, -x^3. \end{aligned} \tag{2.4.1}$$

⁴Notice that any configuration with additional pairs of D4-branes is continuously connected to it. See section 2.5.1 for further discussion. Also, topological A-models related to systems of O4-planes with no stuck D4-brane, and their M-theory interpretation, have appeared in [4], see section 2.5.2 for further discussion.

The geometry is a (Möbius) fiber bundle with base \mathbb{S}^1 , fiber $M_4 \times \mathbb{X}_6$, and structure group \mathbb{Z}_2 .

As before, it is straightforward to add extra D4-brane pairs away from (or on top of) the O4-plane, since they lift to extra M5-brane pairs in M-theory, see section 2.5.1.

2.4.2. M-theory BPS expansion of the real topological string

The M-theory configuration allows for a simple Gopakumar-Vafa picture of amplitudes, which should reproduce the real topological string amplitudes. Since the quotient is acting on the M-theory \mathbb{S}^1 as a half-shift, its effect is not visible locally on the \mathbb{S}^1 . This means that the relevant 5d picture is exactly the same as for the closed oriented setup, c.f. section 2.2, so the relevant BPS states are counted by the standard Gopakumar-Vafa invariants. When compactifying on \mathbb{S}^1 and quotienting by \mathbb{Z}_2 , these states run in the loop as usual, with the only (but crucial) difference that they split according to their parity under the M-theory orientifold action. In the Möbius bundle picture, even components of the original $\mathcal{N} = 2$ multiplets will run on \mathbb{S}^1 with integer KK momentum, whereas odd components run with half-integer KK momentum.

The split is also in agreement with the reduction of supersymmetry by the orientifold, which only preserves 4 supercharges. Note also that the orientifold is not 4d Poincaré invariant, as Lorentz group is broken as

$$\mathrm{SO}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \rightarrow \mathrm{U}(1)_L \times \mathrm{U}(1)_R. \quad (2.4.2)$$

The preserved supersymmetry is not 4d $\mathcal{N} = 1$ SUSY, and in particular it admits BPS particles.

General structure

The states in the Hilbert space \mathcal{H} are groundstates in the SUSY quantum mechanics on the moduli space of wrapped M2-branes. In the orientifold model, these BPS states of the 5d theory fall into two broad classes.

Non-invariant states and the closed oriented contribution

Consider a BPS state $|A\rangle$ associated to an M2 on a curve Σ_g not mapped to itself under the involution σ ; there is an image multiplet $|A'\rangle$ associated to the image curve⁵ Σ'_g . We can now form orientifold even and odd combinations $|A\rangle \pm |A'\rangle$, which run on the \mathbb{S}^1 with integer or half-integer KK momentum, respectively. Since each such pair has $Z_A = Z_{A'}$ and identical multiplet $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ structure and multiplicities (inherited

⁵Actually, the BPS state wavefunctions may be partially supported in the locus in moduli space where the curve and its image combine into an irreducible equivariant curve. Hence, part of the contribution of non-invariant states in this section may spill off to contributions of invariant states discussed later.

from the parent theory), we get the following structure:

$$\begin{aligned}
& \int_{\epsilon}^{\infty} \frac{ds}{s} \sum_{k \in \mathbb{Z}} \frac{1}{4 \sinh^2(\frac{s\lambda}{2})} e^{-sZ} \operatorname{tr} \left\{ \left[(-1)^F e^{-2s\lambda J_3^L - 2\pi i s k} \right] \right. \\
& \quad \left. + \left[(-1)^F e^{-2s\lambda J_3^L - 2\pi i s(k + \frac{1}{2})} \right] \right\} \\
&= \int_{\epsilon}^{\infty} \frac{ds}{s} \frac{1}{4 \sinh^2(\frac{s\lambda}{2})} e^{-sZ} \operatorname{tr} \left[(-1)^{2J_3^L + 2J_3^R} e^{-2s\lambda J_3^L} \right] \sum_{k \in \mathbb{Z}} \left(e^{-2\pi i s k} + e^{-2\pi i s(k + \frac{1}{2})} \right) \\
&= -2 \int_{\epsilon}^{\infty} \frac{ds}{s} n_{\Sigma_g} \left(2i \sinh \frac{s\lambda}{2} \right)^{2g-2} e^{-sZ} \sum_m \delta(s - 2m) \\
&= -2 \sum_{\text{even } p > 0} n_{\Sigma_g} \frac{1}{p} \left(2i \sinh \frac{p\lambda}{2} \right)^{2g-2} e^{-pZ}.
\end{aligned} \tag{2.4.3}$$

The \sinh^{-2} factor corresponds to 4d kinematics, since the orientifold imposes no restriction on momentum in the directions transverse to the fixed locus. We have also denoted n_{Σ_g} the possible multiplicity arising from $SU(2)_R$ quantum numbers. Clearly, because the states are precisely those in the parent $\mathcal{N} = 2$ theory, summing over multiplets reproduces the Gopakumar-Vafa expansion eq. (2.2.12) of the closed oriented contribution to the topological string partition function: this is because even wrappings p on the orientifold \mathbb{S}^1 correspond to both even and odd wrappings on the closed-oriented \mathbb{S}^1 , due to the reduction of the \mathbb{S}^1 by half. This closed oriented contribution must be duly subtracted from the total amplitude, in order to extract the genuine contribution associated to equivariant curves, reproducing the open and unoriented piece of the topological amplitude; this nicely reproduces the subtraction of the closed oriented contribution in the left hand side of eq. (2.3.3).

The conclusion is that contributions with even wrapping belong to the sector of non-invariant states, which heuristically describe disconnected curves in the cover and reproduce the closed oriented topological string.

Invariant states and the open and unoriented contributions

The second kind of BPS states correspond to M2-branes wrapped on curves $\widehat{\Sigma}_{\hat{g}}$ in the cover, mapped to themselves under σ . The overall parity of one such state is determined by the parities of the states in the corresponding $SU(2)_L$ and $SU(2)_R$ representations. We introduce the spaces $\mathcal{H}_{\pm}^{\hat{g}}$ describing the even/odd pieces of the $SU(2)_L$ representation $I_{\hat{g}}$ for fixed \hat{g} . In what follows, we drop the \hat{g} label to avoid cluttering notation. We similarly split the equivariant BPS invariant $\text{GV}'_{\hat{g},\beta}$ (i.e. after removing the pairs of states considered in the previous discussion) into even/odd contributions as

$$\text{GV}'_{\hat{g},\beta} = \text{GV}'_{\hat{g},\beta}^{'+} + \text{GV}'_{\hat{g},\beta}{'-}. \tag{2.4.4}$$

Recalling that states with even/odd overall parity have integer/half-integer KK momenta, we have a structure

$$\begin{aligned}
& \sum_{\beta, \hat{g}} e^{-sZ} \left[\left(\text{GV}'_{\hat{g}, \beta}{}^+ \text{tr}_{\mathcal{H}_+} \mathcal{O} + \text{GV}'_{\hat{g}, \beta}{}^- \text{tr}_{\mathcal{H}_-} \mathcal{O} \right) \sum_{k \in \mathbb{Z}} e^{-2\pi i s k} \right. \\
& \quad \left. + \left(\text{GV}'_{\hat{g}, \beta}{}^+ \text{tr}_{\mathcal{H}_-} \mathcal{O} + \text{GV}'_{\hat{g}, \beta}{}^- \text{tr}_{\mathcal{H}_+} \mathcal{O} \right) \sum_{k \in \mathbb{Z}} e^{-2\pi i s (k + \frac{1}{2})} \right] \\
&= \sum_{\beta, \hat{g}} e^{-sZ} \sum_{m \in \mathbb{Z}} \delta(s - m) \left\{ \left[\text{GV}'_{\hat{g}, \beta}{}^+ \text{tr}_{\mathcal{H}_+} \mathcal{O} + \text{GV}'_{\hat{g}, \beta}{}^- \text{tr}_{\mathcal{H}_-} \mathcal{O} \right] \right. \\
& \quad \left. + (-1)^m \left[\text{GV}'_{\hat{g}, \beta}{}^+ \text{tr}_{\mathcal{H}_-} \mathcal{O} + \text{GV}'_{\hat{g}, \beta}{}^- \text{tr}_{\mathcal{H}_+} \mathcal{O} \right] \right\} \tag{2.4.5} \\
&= \sum_{\beta, \hat{g}} e^{-sZ} \sum_{m \in 2\mathbb{Z}} \delta(s - m) \left(\text{GV}'_{\hat{g}, \beta}{}^+ + \text{GV}'_{\hat{g}, \beta}{}^- \right) \{ \text{tr}_{\mathcal{H}_+} \mathcal{O} + \text{tr}_{\mathcal{H}_-} \mathcal{O} \} \\
& \quad + \sum_{\beta, \hat{g}} e^{-sZ} \sum_{m \in 2\mathbb{Z}+1} \delta(s - m) \left(\text{GV}'_{\hat{g}, \beta}{}^+ - \text{GV}'_{\hat{g}, \beta}{}^- \right) \{ \text{tr}_{\mathcal{H}_+} \mathcal{O} - \text{tr}_{\mathcal{H}_-} \mathcal{O} \}.
\end{aligned}$$

In the next to last line, the traces clearly add up to the total trace over the parent $\mathcal{N} = 2$ multiplet and the GV'^\pm add up to the parent BPS invariants, c.f. eq. (2.4.4). Noticing also that it corresponds to even wrapping contributions $m \in 2\mathbb{Z}$, we realize that this corresponds to a contribution to the closed oriented topological string partition function, c.f. footnote 5. As discussed, it should not be included in the computation leading to the open and unoriented contributions.

The complete expression for the latter is

$$\begin{aligned}
& \sum_{\substack{\beta, \hat{g} \\ m \in 2\mathbb{Z}+1}} \widehat{\text{GV}}_{\hat{g}, \beta} \int_{\epsilon}^{\infty} \frac{ds}{s} \frac{\delta(s - m)}{2 \sinh(\frac{s\lambda}{2})} e^{-sZ} \left\{ \text{tr}_{\mathcal{H}_+} \left[(-1)^{2J_3^L} e^{-s\lambda J_3^L} \right] - \text{tr}_{\mathcal{H}_-} \left[(-1)^{2J_3^L} e^{-s\lambda J_3^L} \right] \right\} \\
&= \sum_{\substack{\beta, \hat{g} \\ \text{odd } m \geq 1}} \widehat{\text{GV}}_{\hat{g}, \beta} \frac{1}{m} \frac{1}{2 \sinh(\frac{m\lambda}{2})} e^{-mZ} \left\{ \text{tr}_{\mathcal{H}_+} \left[(-1)^{2J_3^L} e^{-m\lambda J_3^L} \right] - \text{tr}_{\mathcal{H}_-} \left[(-1)^{2J_3^L} e^{-m\lambda J_3^L} \right] \right\}, \tag{2.4.6}
\end{aligned}$$

where we have introduced the integers, which we call real BPS invariants,

$$\widehat{\text{GV}}_{\hat{g}, \beta} := \text{GV}'_{\hat{g}, \beta}{}^+ - \text{GV}'_{\hat{g}, \beta}{}^-. \tag{2.4.7}$$

These integer numbers $\widehat{\text{GV}}$'s are those playing the role $\widetilde{\text{GV}}$'s in eq. (2.3.3). Note however that their correct physical interpretation differs from that in [91], where they were rather identified as our $\text{GV}'_{\hat{g}, \beta}$. Note also that the correct invariants eq. (2.4.7) are equal mod 2 to the parent $\text{GV}_{\hat{g}, \beta}$, proposed in [91], just like the $\text{GV}'_{\hat{g}, \beta}$.

In eq. (2.4.6) we have taken into account that these states, being invariant under the orientifold, propagate only in the 2d fixed subspace of the 4d spacetime, resulting in a single power of $(2 \sinh)$ in the denominator. This also explains the factor of 2 in the graviphoton coupling relative to eq. (2.2.4). In the next section we fill the gap of showing the promised equality of the even and odd multiplicities, and compute the trace difference in the last expression.

Jacobian and computation of $SU(2)_L$ traces

We must now evaluate the trace over the even/odd components of the Hilbert space of a parent $\mathcal{N} = 2$ BPS multiplet. This is determined by the parity of the corresponding zero modes on the particle worldline. As reviewed in section 2.2, the traces are non-trivial only over the cohomology of the Jacobian of $\widehat{\Sigma}_{\hat{g}}$ which determines the $SU(2)_L$ representation. We now focus on its parity under the orientifold.

Consider for example the case of I_1 , c.f. eq. (2.2.7). We introduce the formal split of the trace into traces over \mathcal{H}_{\pm}

$$t_1 = t_1^+ \ominus t_1^-, \quad (2.4.8)$$

where \pm denotes orientifold behavior and \ominus denotes a formal combination operation, which satisfies $\ominus^2 = 1$ (it corresponds to the $(-1)^m$ factor once the wrapping number m has been introduced, c.f. eq. (2.4.5)). Since the orientifold action is an antiholomorphic involution on the worldsheet, it acts as $dz \leftrightarrow d\bar{z}$, so eq. (2.2.7) splits as

$$I_1 = \left(\frac{1}{2}\right) \oplus 2(0) = \begin{pmatrix} + \\ - \end{pmatrix} \oplus (+) \oplus (-), \quad (2.4.9)$$

which gives a trace

$$t_1 = \underbrace{(1 - e^{s/2})}_{t_1^+} \ominus \underbrace{(1 - e^{-s/2})}_{t_1^-}, \quad (2.4.10)$$

where, to avoid notational clutter, we have reabsorbed λ into s .

Since the creation and annihilation operators associated to different 1-forms commute, the argument generalizes easily to higher genus, and the trace over a representation $I_{\hat{g}}$ has the structure

$$t_{\hat{g}} = (t_1^+ \ominus t_1^-)^{\hat{g}} = t_{\hat{g}}^+ \ominus t_{\hat{g}}^-, \quad (2.4.11)$$

where $t_{\hat{g}}^+$ and $t_{\hat{g}}^-$ contain even and odd powers of t_1^- , respectively. For instance, for I_2 we have to trace over

$$\begin{pmatrix} + \\ - \\ + \end{pmatrix} \oplus 2 \begin{pmatrix} + \\ - \end{pmatrix} \oplus 2 \begin{pmatrix} - \\ + \end{pmatrix} \oplus 2(+) \oplus 3(-) \quad (2.4.12)$$

and obtain

$$t_2 = \underbrace{(t_1^+)^2 + (t_1^-)^2}_{2+e^{-s}+e^s-2e^{-s/2}-2e^{s/2}} \ominus \underbrace{2t_1^+t_1^-}_{4-2e^{-s/2}-2e^{s/2}} = (t_1^+ \ominus t_1^-)^2 = t_2^+ \ominus t_2^-. \quad (2.4.13)$$

We are now ready to compute the final expression for the BPS expansion

The BPS expansion

Recalling eq. (2.3.3), the genuine open and unoriented contribution reduces to the odd wrapping number case eq. (2.4.6). Interestingly, the trace difference can be written (restoring the λ)

$$t_{\hat{g}}^+ - t_{\hat{g}}^- = \left(-2 \sinh \frac{s\lambda}{2}\right)^{\hat{g}}. \quad (2.4.14)$$

This is clear from eqs. (2.4.10) and (2.4.13) for I_1 , I_2 respectively, and holds in general.

The final result for the BPS amplitude, which corresponds to the BPS expansion of the open and unoriented partition function, is

$$\sum_{\substack{\beta, \hat{g} \\ \text{odd } m \geq 1}} \widehat{\text{GV}}_{\hat{g}, \beta} \frac{1}{m} \left[2 \sinh \left(\frac{m\lambda}{2} \right) \right]^{\hat{g}-1} e^{-mZ}. \quad (2.4.15)$$

This has the precise structure to reproduce the conjecture in [91] as in eq. (2.3.3), with the invariants defined by eq. (2.4.7). In particular we emphasize the nice matching of exponents of the sinh factors (achieved since for the covering $\hat{g} - 1 = \chi$) and of the exponential $e^{-Z} = q^{d/2}$ for a one-modulus \mathbb{X}_6 (the factor of $1/2$ coming from the volume reduction due to the \mathbb{Z}_2 quotient.)

The only additional ingredient present in eq. (2.3.3) is the restriction on the degree, which is related to the conjectured tadpole cancellation condition, and which also admits a natural interpretation from the M-theory picture, as we show in the next section. We simply advance that this restriction applies to examples with $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$. Our formula above is the general BPS expansion of the real topological string on a general CY threefold.

We anticipate that, once the tadpole cancellation discussed below is enforced, our derivation of eq. (2.4.15) provides the M-theory interpretation for the integer quantities $\widehat{\text{GV}}$ appearing in eq. (2.3.3) as conjectured in [91]. Therefore the real topological string is computing (weighted) BPS multiplicities of equivariant M2-brane states in M-theory, with the weight given by an orientifold parity sign, c.f. eq. (2.4.7). It would be interesting to perform a computation of the numbers appearing in eq. (2.4.15) along the lines of [48, 61].

2.4.3. Tadpole cancellation

In this section we discuss the M-theory description of the tadpole cancellation condition, in examples of the kind considered in the literature, i.e. with $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$ and $H_2(\mathbb{X}_6, L; \mathbb{Z}) = \mathbb{Z}$ (like the quintic or local \mathbb{CP}^2), for which one trades the class β for the degree $d \in \mathbb{Z}$. The argument involves several steps.

First step: Restriction to even degree

Consider the relative homology exact sequence

$$\begin{array}{ccccc} H_2(\mathbb{X}_6; \mathbb{Z}) & \xrightarrow{2} & H_2(\mathbb{X}_6, L; \mathbb{Z}) & \rightarrow & H_1(L; \mathbb{Z}). \\ \parallel & & \parallel & & \parallel \\ \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}_2 \end{array} \quad (2.4.16)$$

Since (the embedded image of) a crosscap doesn't intersect the lagrangian L , its class must be in the kernel of the second map, i.e. the image of the first. Thus, every crosscap contributes an even factor to the degree d . For boundaries, the same argument implies

that boundaries wrapped on an odd multiple of the non-trivial generator of $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$ contribute to odd degree, while those wrapped on an even multiple of the \mathbb{Z}_2 1-cycle contribute to even degree. This restricts the possible cancellations of crosscaps to even degree boundaries.

Second step: Relative signs from background form fields

We now show that there is a relative minus sign between crosscaps and disks associated to the same (necessarily even degree) homology class. As mentioned in section 2.4.1, the M-theory lift contains a background 3-form C_3 along the 3-cycles $(\mathbb{CP}^1 \times \mathbb{S}^1)/\mathbb{Z}_2$, with the \mathbb{Z}_2 acting antiholomorphically over \mathbb{CP}^1 ; this corresponds to a half-unit of NSNS 2-form flux on any crosscap \mathbb{RP}^2 surrounding the O4-plane in the type IIA picture. In M-theory, the reduction of C_3 along the \mathbb{CP}^1 produces a 5d gauge boson, under which any M2-brane is charged with charge c , where c is the number of crosscaps in the embedded curve in \mathbb{X}_6/σ . The 3-form background corresponds to a non-trivial \mathbb{Z}_2 Wilson line turned on along the M-theory \mathbb{S}^1 , and produces an additional contribution to the central charge term Z , which (besides the KK term) reads

$$Z = dt + \frac{i}{2}c. \quad (2.4.17)$$

Once we exponentiate, and perform Poisson resummation eq. (2.2.11), this gives a contribution $(-1)^{m \times c}$, with m the wrapping number, which is odd for the genuine equivariant contributions. This extra sign does not change the contributions of curves with even number of crosscaps. Note that the above also agrees with the fact that the (positive) number of crosscaps is only defined mod 2.

In contrast, boundaries do not receive such contribution,⁶ and therefore there is a relative sign between contributions from curves which fall in the same homology class, but differ in trading a crosscap for a boundary.

Third step: Bijection between crosscaps and boundaries

To complete the argument for the tadpole cancellation condition, one needs to show that there is a one-to-one correspondence between curves which agree except for a replacement of one crosscap by one boundary. The replacement can be regarded as a local operation on the curve, so the correspondence is a bijection between disk and crosscap contributions.

More precisely we want to show that for every homologically trivial disk which develops a node on top of L we can find a crosscap, and viceversa. This is mathematically a nontrivial statement, for which we weren't able to find an explicit construction going beyond the local model of eq. (2.A.3). Moreover, in higher genera this problem has not been tackled by mathematicians yet. Nonetheless, following [88], we propose a model

⁶Note that odd degree boundaries can receive an extra sign due to a possible \mathbb{Z}_2 Wilson line on the D4-brane. This however does not affect the even degree boundaries, which are those canceling against crosscaps. Hence it does not have any effect in the explanation of the tadpole cancellation condition.

for gluing boundaries on the moduli space in the genus zero case, which points towards the desired bijection. The original argument applies to holomorphic maps, relevant to Gromov-Witten invariants; we expect similar results for holomorphic embeddings, relevant for Gopakumar-Vafa invariants.

The main point is that integrals over moduli spaces of Riemann surfaces make sense and are independent of the choice of complex structure if the moduli space has a virtually orientable fundamental cycle without real codimension one boundaries (RCOB). If $L \neq \emptyset$, in order to achieve this, one has to consider together contributions coming roughly speaking from open and unoriented worldsheets, as proposed by [91].

We are interested in elements of RCOB in which a piece of the curve degenerates as two spheres touching at a point q :

$$(f, \Sigma = \Sigma_1 \cup_q \Sigma_2), \quad (2.4.18)$$

where f is the holomorphic map, $\Sigma_i = \mathbb{CP}^1$, the involution exchanges the Σ_i and $f(q) \in L$. For real $\epsilon \neq 0$ one can glue Σ into a family of smooth curves, described locally as⁷

$$\Sigma_\epsilon = \{(z, w) \in \mathbb{C}^2 : zw = \epsilon\}. \quad (2.4.19)$$

For $\epsilon \in \mathbb{R}$, Σ_ϵ inherits complex conjugation from eq. (2.4.20), and the fixed point set is \mathbb{S}^1 if $\epsilon > 0$, empty if $\epsilon < 0$. They correspond to an equivariant curve with two different involutions, which in terms of homogeneous coordinates on \mathbb{CP}^1 , can be described as

$$(u : v) \mapsto (\bar{u} : \pm \bar{v}), \quad (2.4.20)$$

leading to either a boundary or a crosscap. The RCOB corresponding to sphere bubbling in the ‘+’ case is the same as the RCOB for the ‘−’ case. By attaching them along their common boundary, we obtain a moduli space whose only RCOB corresponds to disk bubbling. The resulting combined moduli space admits a Kuranishi structure and produces well-defined integrals.

Final step

Using the above arguments, we can now derive eq. (2.3.1), as follows. First, the tadpole cancellation removes contributions where the number of crosscaps c is odd, so taking $\chi = 2g - 2 + h + c$ we have $\chi \equiv h \pmod{2}$, where h denotes the number of boundaries. Second, the value of $d = \sum d_i \pmod{2}$ can only get contributions from boundaries and crosscaps (since contributions of pieces of the curve away from L cancel mod 2 from the doubling due to the orientifold image); moreover contributions from crosscaps and even degree boundaries cancel. Hence, the only contributions arise from boundaries with odd terms d_i , so clearly $d \equiv h \pmod{2}$. We hence recover eq. (2.3.1).

⁷In this equivariant covering picture, we only consider singularities of type (1) as in Definition 3.4 of [69], since the covering does not have boundaries.

2.5. Extensions and relations to other approaches

2.5.1. Adding extra D4-brane pairs

The discussion in the previous sections admits simple generalizations, for instance the addition of N extra D4-brane pairs in the type IIA picture.⁸ The two branes in each pair are related by the orientifold projection, but can otherwise be placed at any location, and wrapping general lagrangian 3-cycles in \mathbb{X}_6 . For simplicity, we consider them to wrap the O4-plane lagrangian 3-cycle L , and locate them on top of the O4-plane in the spacetime dimensions as well.

In the M-theory lift, we have the same quotient acting as a half shift on the M-theory \mathbb{S}^1 (times the antiholomorphic involution of \mathbb{X}_6 and the spacetime action $x^2, x^3 \rightarrow -x^2, -x^3$), now with the extra D4-brane pairs corresponding to extra M5-brane pairs, related by the M-theory orientifold symmetry [47]. Notice that since the orientifold generator is freely acting, there is no singularity, and therefore no problem in understanding the physics associated to these M5-branes. In the 5d theory, there is no orientifold, and the introduction of the M5-branes simply introduces sectors of BPS states corresponding to open M2-branes wrapped on holomorphic 2-chains with boundary on the M5-brane lagrangians. Their multiplicity is precisely given by the open oriented Gopakumar-Vafa invariants [77]. These particles run in the M-theory \mathbb{S}^1 and must be split according to their parity under the orientifold action, which determine the appropriate KK momentum quantization. By the same arguments as in section 2.4.2, the contributions which have even wrapping upon Poisson resummation actually belong to the open oriented topological amplitude, and should be discarded. To extract the genuine open unoriented amplitude, we must focus on surfaces mapped to themselves under σ , and restrict to odd wrapping number. In analogy with section 2.4.2 and [77], the amplitude can be written

$$\sum_{\text{odd } m \geq 1} \sum_{\beta, r, \mathcal{R}} \frac{N_{\beta, r, \mathcal{R}}^+ - N_{\beta, r, \mathcal{R}}^-}{2m \sinh(\frac{m\lambda}{2})} e^{-m\beta \cdot t - mr\lambda} \text{tr}_{\mathcal{R}} \prod_{i=1}^{b_1(L)} V_i^m. \quad (2.5.1)$$

The sum in m runs only over positive odd integers. The $N_{\beta, \mathcal{R}, r}^{\pm}$ are the multiplicities of (even or odd) states from M2-branes on surfaces in the class β , with spin r under the rotational $U(1)$ in the 01 dimensions,⁹ and in the representation \mathcal{R} of the background brane $SO(2N)$ symmetry. The V_i denote the lagrangian moduli describing the $SO(2N)$ Wilson lines (complexified with deformation moduli), possibly turned on along the non-trivial 1-cycles of L . Since the presence of the M5-branes breaks the $SU(2)_L$ structure, it is not possible to perform a partial sum over such multiplets explicitly.

A clear expectation from the type IIA perspective is that the total amplitude (namely adding the original contribution in the absence of D4-brane pairs) could be rewritten to display an $SO(2N + 1)$ symmetry, combining the stuck and paired D4-branes. This is

⁸The addition of one extra unpaired D4-brane would modify drastically the M-theory lift of the configuration, as discussed in the next section.

⁹Actually, it is the charge under the $U(1)_L \subset SU(2)_L$, which describes the coupling to the self-dual graviphoton background.

possible thanks to the close analogy of the above expression with eq. (2.4.15), once we expand the contribution $\sinh^{\hat{g}}$ to break down the $SU(2)_L$ multiplet structure. By suitably subtracting contributions $N^+ - N^-$ to the multiplicities \widehat{GV} , one can expect to isolate the $SO(2N + 1)$ symmetric contribution.

For instance, take the case of only one matrix V (e.g. $H_1(L; \mathbb{Z}) = \mathbb{Z}$ or \mathbb{Z}_2). Combining eq. (2.5.1) with eq. (2.4.15), we have

$$\sum_{\beta, r, m} \frac{e^{-mZ}}{2m \sinh(\frac{m\lambda}{2})} e^{-mr\lambda} \left[\widehat{GV}_{r, \beta} + \left(N_{\beta, r, \mathcal{R}}^+ - N_{\beta, r, \mathcal{R}}^- \right) \text{tr}_{\mathcal{R}} V^m \right], \quad (2.5.2)$$

where we have introduced $\widehat{GV}_{r, \beta}$ as the combination of real BPS invariants $\widehat{GV}_{\hat{g}, \beta}$ describing the multiplicity of M2-brane states with 2d $U(1)$ spin r . For a given representation \mathcal{R} , the requirement that the expression in square brackets combines into traces of $SO(2N + 1)$ seems to imply non-trivial relations between the open and real BPS invariants for different representations of $SO(2N)$. It would be interesting to study these relations further.

2.5.2. Relation to other approaches

In this section we describe the relation of our system with other unoriented A-model topological strings and their physical realization in M-theory.

M-theory lift of the four O4-planes

As discussed in [35, 47] there are four kinds of O4-planes in type IIA string theory, with different lifts to M-theory. We describe them in orientifolds of type IIA on $\mathbb{X}_6 \times M_4$, with the geometric part of the orientifold acting as an antiholomorphic involution on \mathbb{X}_6 and $x^2, x^3 \rightarrow -x^2, -x^3$ on the 4d spacetime.

- An $O4^-$ -plane (carrying -1 units of D4-brane charge, as counted in the covering space) with no D4-branes on top. Its lift to M-theory is a geometric orientifold $M_2 \times (\mathbb{R}^2 \times \mathbb{X}_6)/\mathbb{Z}_2 \times \mathbb{S}^1$. Inclusion of additional D4-brane pairs (in the covering space) corresponds to including additional M5-brane pairs in the M-theory lift.
- An $O4^0$ -plane, which can be regarded as an $O4^-$ with one stuck D4-brane. We recall that its M-theory lift, exploited in this chapter, is $M_2 \times (\mathbb{R}^2 \times \mathbb{X}_6 \times \mathbb{S}^1)/\mathbb{Z}_2$, with the \mathbb{Z}_2 including a half-period shift of the \mathbb{S}^1 . Additional D4-brane pairs correspond to additional M5-brane pairs, as studied in the previous section.
- An $O4^+$ -plane (carrying $+1$ units of D4-brane charge). Its lift to M-theory is a geometric orientifold $M_2 \times (\mathbb{R}^2 \times \mathbb{X}_6)/\mathbb{Z}_2 \times \mathbb{S}^1$ with two stuck M5-branes on top. The M5-branes are stuck because they do not form an orientifold pair, due to a different worldvolume Wilson line [35].
- An $\widetilde{O4}^+$ -plane, which can be regarded as an $O4^+$ with an extra RR background field. Its M-theory lift is our $M_2 \times (\mathbb{R}^2 \times \mathbb{X}_6 \times \mathbb{S}^1)/\mathbb{Z}_2$ geometry, with one stuck M5-brane fixed by the \mathbb{Z}_2 action.

The $O4^-$ vs. the $O4^0$ case

Several references, e.g. [1, 4, 20, 21, 81], consider unoriented A-models with no open string sector, corresponding to an M-theory lift $(M_4 \times \mathbb{X}_6)/\mathbb{Z}_2 \times \mathbb{S}^1$. This corresponds to the case of the $O4^-$ -plane (for their relation with the $O4^+$ case, see later). The key difference with our setup is that in general there are \mathbb{Z}_2 fixed points, which correspond to $L \times \mathbb{R}^2 \times \mathbb{S}^1$. The physics near these singularities cannot be addressed with present technology. However, since crosscap embeddings do not intersect L , it is possible to meaningfully propose an M-theory Gopakumar-Vafa interpretation of the unoriented topological string amplitude. This can also be extended to open string sectors, as long as the D4-branes (or M5-branes in the M-theory lift) are introduced in pairs and kept away from the singular locus. As we discuss later on, this limits the possibility of reproducing the right physics for the $O4^+$.

A second difference from our setup is that the orientifold action in M-theory is felt even locally on the \mathbb{S}^1 , i.e. already at the level of the 5d theory, and breaks the $SU(2)_L$ symmetry. Therefore the structure of multiplets need not correspond to full $SU(2)_L$ multiplets, although this is explicitly assumed in most of these references. Although supported by the appropriate integrality properties derived from the analysis, these extra assumptions obscure the physical derivation of the BPS integrality structures.

We emphasize again that these properties differ in our system, which corresponds to the lift of the $O4^0$ -plane. The $SU(2)_L$ multiplet structure is directly inherited from the parent theory, and is therefore manifestly present, without extra assumptions.

The $O4^+$ case

The case of the $O4^+$ -plane has been discussed in the literature as a minor modification of the $O4^-$ case. Indeed, from the viewpoint of the type IIA theory (equiv. of the topological A-model), both systems are related by a weight $(-1)^c$ for any worldsheet amplitude with c crosscaps (corresponding to a change in the NSNS 2-form background around the orientifold plane). This motivates an immediate BPS invariant expansion of the unoriented A-model amplitude corresponding to the $O4^+$, see e.g. [1, 4, 81].

On the other hand, the actual M-theory lift of the $O4^+$ -plane corresponds not to the geometry $M_2 \times (\mathbb{R}^2 \times \mathbb{X}_6)/\mathbb{Z}_2 \times \mathbb{S}^1$ with a different choice of 3-form background, but rather to the same geometry as the $O4^-$ -plane, with the addition of two stuck M5-branes at the \mathbb{Z}_2 fixed point. In this lift, interestingly, the M-theory picture contains both unoriented and open M2-brane curves, which should combine together to reproduce a purely unoriented Gromov-Witten worldsheet expansion; the latter moreover admits a BPS expansion in terms of purely closed M2-brane curves, up to some sign flips. It is non-trivial to verify how these pictures fit together, in particular given the difficulties in dealing with M2-branes ending on M5-branes stuck at the \mathbb{Z}_2 fixed point in the M-theory geometry. The details of this connection are therefore still open.

The $\widetilde{\text{O4}}^+$ case

Finally, the case of the $\widetilde{\text{O4}}^+$ -plane has not been considered in the literature. Actually, it is closely related to the lift of the O4^0 , with the addition of one M5-brane. It is therefore very similar to the systems in the previous section, and the corresponding amplitude is essentially given by eq. (2.5.1), for $2N + 1$ M5-branes (allowing for the addition of N brane pairs).

In this case there is also an interesting interplay with the type IIA picture, although in the opposite direction as compared with the O4^0 case. Namely, the M-theory lift contains one more brane than the corresponding type IIA picture. This implies that in the BPS expansion both closed and open M2-brane states have to combine together to reproduce the crosscap worldsheet diagram in type IIA. It would be interesting to carry out this comparison further, although this may be difficult due to the presence of a non-trivial RR background in the type IIA orientifold, which may render the worldsheet analysis difficult.

2.6. Conclusions and open issues

We have discussed the BPS integer expansion of the real topological string in [91], using the M-theory lift of the O4 -plane with one stuck D4-brane. Since the geometry is a \mathbb{Z}_2 quotient acting freely in the M-theory \mathbb{S}^1 , the 5d setup enjoys an enhancement to 8 supercharges and is identical to that in the closed oriented Gopakumar-Vafa system. The subtleties due to the orientifold quotient arise as a compactification effect modifying the KK momentum of the BPS states on the \mathbb{S}^1 according to their parity under the orientifold action. This allows for a clean derivation of the BPS integer expansion, without the extra assumptions that pop up in other unoriented A-models.

Although we recover the BPS expansion conjectured in [91], our derivation shows the correct identification of the BPS invariants not as the equivariant sector of the parent Gopakumar-Vafa invariants, but rather a weighted version thereof.

The M-theory picture provides a complementary viewpoint on the sign choices implied by the tadpole cancellations in models where the fixed lagrangian 3-cycle L has $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$ [91]. More in general, the BPS integer expansion we propose is valid for other situations, providing a general definition of the real topological string.

The careful M-theory lift of other O4 -planes suggests non-trivial relations between their BPS invariant expansions, for instance the addition of an open M2-brane sector (associated to two stuck M5-branes) to the lift of the O4^- -plane should reproduce a sign flip in odd crosscap contributions. This seems to imply non-trivial relations among the unoriented and open BPS invariants in M-theory orientifolds with fixed points. We hope to return to these and other questions in the future.

2.A. Review of Walcher's real topological string

We now give a short review of A-model localization as in [91]. Take for concreteness Fermat quintic, given by

$$\left\{ \sum_{i=1}^5 x_i^5 = 0 \right\} \subset \mathbb{CP}^4 \quad (2.A.1)$$

and involution

$$\sigma : (x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (\bar{x}_2 : \bar{x}_1 : \bar{x}_4 : \bar{x}_3 : \bar{x}_5) \quad (2.A.2)$$

which gives a fixed point lagrangian locus L with \mathbb{RP}^3 topology, hence $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$.

2.A.1. Tadpole cancellation in the topological string

Requiring a function f to be equivariant implies that fixed points of Ω are mapped to L , but one has to further specify their homology class in $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$. When that class is trivial, then under deformation of the map it can happen that the boundary is collapsed to a point on L .¹⁰

The local model for this phenomenon is a Veronese-like embedding $\mathbb{CP}^1 \rightarrow \mathbb{CP}^2$ defined by a map $(u : v) \mapsto (x : y : z)$ depending on a target space parameter a , concretely

$$x = au^2, \quad y = av^2, \quad z = uv. \quad (2.A.3)$$

The image can be described as the conic $xy - a^2z^2 = 0$, and it is invariant under σ if $a^2 \in \mathbb{R}$. The singular conic $a = 0$ admits two different equivariant smoothings, determined by the nature of a :

$$\begin{array}{lll} a \in \mathbb{R} & (u : v) \sim (\bar{v} : \bar{u}) & \text{disk} \\ a \in i\mathbb{R} & (u : v) \sim (\bar{v} : -\bar{u}) & \text{crosscap.} \end{array} \quad (2.A.4)$$

The proposal of [91] to account for this process is to count disks with collapsible boundaries and crosscaps together. Specifically, there is a one-to-one correspondence between even degree maps leading to boundaries and maps leading to crosscaps (which must be of even degree, in order to be compatible with the antiholomorphic involution, as already manifest in the above local example). The tadpole cancellation condition amounts to proposing the combination of these paired diagrams, such that certain cancellations occur. For instance, the amplitude $\mathcal{R}^{(g,h)}$ (odd number of crosscaps) vanish (due to the cancellation of the unpaired crosscap with an odd degree boundary). Similarly, for the remaining contributions, in terms of χ and d , the tadpole cancellation imposes the restriction

$$d \equiv h \equiv \chi \pmod{2}. \quad (2.A.5)$$

The first equality follows from the requirement that odd degree contributions only come from boundaries (homologically trivial, i.e. even degree, ones cancel against crosscaps), while the second from the requirement that there be no unpaired crosscaps.

¹⁰This is a real codimension one stratum in the moduli space, as discussed in section 2.3.

2.A.2. Rules of computation

One can then *postulate* the existence of a well-defined fundamental class allowing to integrate over the moduli space $\mathcal{M} = \overline{\mathcal{M}}_\Sigma(\mathbb{CP}^4, d)$ of maps, defined as the top Chern class of an appropriate bundle \mathcal{E}_d over \mathcal{M} , whose fiber is given by $H^0(\Sigma, f^*\mathcal{O}(5))$:

$$\widetilde{\text{GW}}_d^\Sigma = \int_{\mathcal{M}} \mathbf{e}(\mathcal{E}_d). \quad (2.A.6)$$

For the integral to make sense, dimensions are constrained as

$$5d + 1 = \dim \mathcal{E}_d \stackrel{!}{=} \text{vdim } \overline{\mathcal{M}}_{g,n}(\mathbb{CP}^D, \beta) = c_1 \cdot \beta + (3 - D)(g - 1) + n, \quad (2.A.7)$$

where $\beta \in H_2(\mathbb{CP}^D; \mathbb{Z})$ is to be identified with d in this particular case, and n denotes the number of punctures.

The next step is to apply Atiyah-Bott localization to the subtorus $\mathbb{T}^2 \subset \mathbb{T}^5$ compatible with σ . As explained in [64], the fixed loci of the torus action are given by nodal curves, in which any node or any component of non-zero genus is collapsed to one of the fixed points in target space, and any non-contracted rational component is mapped on one of the coordinate lines with a standard map of given degree d_i :

$$f(w_1 : w_2) = (0 : \dots : 0 : w_1^{d_i} : 0 : \dots : 0 : w_2^{d_i} : 0 : \dots : 0). \quad (2.A.8)$$

The components of the fixed locus can be represented by a decorated graph Γ and one has well-defined rules for associating a graph to a class of stable maps.

For the case of real maps, one has to be extra careful and require the decoration to be compatible with the action of Ω and σ . For example, consider a fixed edge: if we think of $z = w_1/w_2$, then in eq. (2.A.8) $z \mapsto 1/\bar{z}$ is compatible (i.e. f is equivariant) with any degree, while $z \mapsto -1/\bar{z}$ requires even degree, because our involution acts on the target space as $(x_1 : x_2 : \dots) \mapsto (\bar{x}_2 : \bar{x}_1 : \dots)$ i.e. $w = x_1/x_2 \mapsto 1/\bar{w}$.

A careful analysis in [67, 78, 90, 91] allows to conclude that the localization formula takes the form

$$\widetilde{\text{GW}}_d^\Sigma = (-1)^{p(\Sigma)} \sum_{\Gamma} \frac{1}{\text{Aut } \Gamma} \int_{\mathcal{M}_\Gamma} \frac{\mathbf{e}(\mathcal{E}_d)}{\mathbf{e}(\mathcal{N}_\Gamma)}, \quad (2.A.9)$$

where the following prescriptions are used:

- (I) the $(-1)^{p(\Sigma)}$ factor in front of the localization formula is put by hand in order to fix the relative orientation between different components of moduli space;
- (II) for any fixed edge of even degree, the homologically trivial disk and crosscap contributions are summed, with a relative sign such that they cancel. This is the above mentioned *tadpole cancellation*.

Finally, [91] proposes an integer BPS interpretation of the obtained rational numbers $\widetilde{\text{GW}}$: by combining them with the above prescribed signs at fixed χ , one gets integer numbers $\widetilde{\text{GV}}$, which are conjectured to reproduce a BPS expansion for the open-unoriented topological amplitudes.

2.B. Physical couplings

An interesting question one can ask is what kind of coupling the topological string is computing in the IIA physical theory. This has a clear answer for the closed oriented sector [9, 17], while some proposals have been made for open oriented [77] and unoriented sectors [81]. Here we make a proposal for the analogous expression in our unoriented model.

We have 4 supercharges in $1 + 1$ dimensions, and we'd like to find a good splitting of the $\mathcal{N} = 2$ Weyl tensor

$$\mathcal{W}_{\mu\nu}^{ij} = T_{\mu\nu}^{ij} + R_{\mu\nu\rho\lambda}\theta^i\sigma^{\rho\lambda}\theta^j + \dots, \quad (2.B.1)$$

where one requires the graviphoton field strength T to acquire a self-dual background, and hence can write

$$\mathcal{W}_{\alpha\beta} = \frac{1}{2}\varepsilon_{ij}\mathcal{W}_{\mu\nu}^{ij}(\sigma^\mu)_{\alpha\dot{\alpha}}(\sigma^\nu)_{\beta\dot{\beta}}\varepsilon^{\dot{\alpha}\dot{\beta}}. \quad (2.B.2)$$

A natural guess is that the amplitude $\mathcal{G}^{(\chi)}$ as in eq. (2.3.2) computes

$$\int d^4x \int d^4\theta \delta^2(\theta)\delta^2(x) \left(\mathcal{G}^{(\chi)}(t) - \frac{1}{2}\mathcal{F}^{(g_\chi)}(t) \right) (\mathcal{W} \cdot v)^{\hat{g}}, \quad (2.B.3)$$

where $\mathcal{W} \cdot v = \mathcal{W}_{\alpha\beta}\sigma_{\alpha\beta}^{\mu\nu}v_{\mu\nu}$. This has a contribution $RT^{\hat{g}-1}$ (in the covering picture $\hat{g}-1 = \chi$) which in principle can generate (via SUSY) the \sinh^{-1} power in the Schwinger computation, taking into account the fact that the orientifold halves the number of fermion zero modes on the Riemann surface.

It would be interesting to discuss the appearance of this contribution from different topologies at fixed χ .

Chapter 3

Towards a gauge theory interpretation

3.1. Introduction

A most interesting connection between gauge theory and string theory is the relation between non-perturbative instanton corrections in 4d/5d gauge theories with 8 supercharges [74, 76], and the topological string partition functions on non-compact toric Calabi-Yau (CY) three-folds \mathbb{X} (in their formulation in terms of BPS invariants [36, 37, 77]). Basically, the BPS M2-branes on compact 2-cycles of \mathbb{X} are instantons of the gauge theory arising from M-theory on \mathbb{X} .

We are interested in extending this correspondence to systems with orientifold projections. A natural starting point is the real topological string, introduced in [91] and studied in compact examples in [66] and in non-compact CY three-folds in [65, 67]. This real topological string is physically related to type IIA on a CY three-fold \mathbb{X} quotiented by an orientifold, given by an antiholomorphic involution σ on \mathbb{X} and a flip on two of the 4d spacetime coordinates (say x^2, x^3); it thus introduces an O4-plane, spanning the lagrangian 3-cycle given by the fixed point set,¹ and the two fixed 4d spacetime dimensions x^0, x^1 . In addition, the topological tadpole cancellation [91] requires the introduction of a single stuck D4-brane on top of the (negatively charged) O4-plane, producing local cancellation of RR charge.

The M-theory lift of this type IIA configuration [79] corresponds to a freely acting quotient of M-theory on $\mathbb{X} \times \mathbb{S}^1$, in which the action σ on \mathbb{X} (and the flip of two 4d coordinates) is accompanied by a half-period shift along the \mathbb{S}^1 . This M-theory lift provides a reinterpretation of the real topological string partition function in terms of real BPS invariants, which are essentially given by a combination of the parent Gopakumar-Vafa (GV) BPS invariants, weighted by the ± 1 eigenvalue of the corresponding state under the orientifold action.

This orientifolding can be applied in the context of type IIA/M-theory on non-compact toric CY three-folds \mathbb{X} which realizes 5d gauge theories. More specifically, one should consider the 5d gauge theory compactified to 4d, with an orientifold acting non-trivially as a shift on the \mathbb{S}^1 . Since the orientifold plane is real codimension 2 in the 4d Minkowski dimensions, the system describes the gauge theory in the presence of a surface defect. Certain surface operators have been studied in [30, 39, 41, 42, 57], also in the context of M-theory/gauge theory correspondence, describing them by the introduction of D2-branes/M2-branes in the brane setup or D4-branes in the geometric engineering [28, 77, 87]. An important difference with our discussion is that the holonomy of our surface operators is an outer automorphism of the original gauge group. In our case, the properties of the gauge theory in this orientifold background are implicitly defined by the real topological vertex, even though they should admit an eventual intrinsic gauge-theoretical description. Hence, one can regard our real topological string results as a first step in the study of a novel kind of gauge theory surface operators.

Our strategy is as follows:

- For concreteness, we focus on the geometry which realizes a pure $SU(N)$ gauge

¹We focus on cases with non-trivial fixed point sets; freely acting orientifolds could be studied using similar ideas.

theory, and other similarly explicit examples. We construct the real topological string on non-compact toric CY three-folds \mathbb{X} by using the real topological vertex formalism [65].

- In the M-theory interpretation, the real topological string amplitudes correspond to a one-loop diagram of a set of 5d BPS particles from wrapped M2-branes, suitably twisted by the orientifold action as they propagate on the \mathbb{S}^1 . An important point is that the effect of the orientifold action arises only after the compactification to 4d on \mathbb{S}^1 , so the 5d picture is identical to the parent theory. Therefore, the correspondence between M-theory and 5d $\mathcal{N} = 1$ gauge theory is untouched.
- The corresponding statement on the gauge theory side is that the 4d partition function of the gauge theory in the presence of the orientifold surface defect must be given by the compactification of the original 5d gauge theory on \mathbb{S}^1 , but with modified periodic/anti-periodic boundary conditions for fields which are even/odd under the orientifold action. This can in principle be implemented as the computation of the Witten index with an extra twist operator in the trace. This kind of operator has not appeared in the literature. We use the comparison of the oriented and real topological vertex partition functions to better understand the nature and action of this operator on the gauge theory.

Even though we do not achieve a completely successful gauge theoretical definition of the orientifold operation, we obtain a fairly precise picture of this action in some concrete situations. Moreover, our discussion of the topological vertex amplitudes reveals new properties in the unoriented case.

3.2. Review of the oriented case

In this section we review the correspondence between BPS M2-brane invariants of M-theory on a toric CY three-fold singularity \mathbb{X} and the supersymmetric gauge theory Nekrasov partition function [29, 46, 53, 54, 86, 94]. We will take advantage to introduce useful tools and notations to be used in the discussion of the orientifolded case.

3.2.1. The topological string

BPS expansion of the topological string

We start with a brief review of closed oriented topological string interpreted in terms of BPS states in M-theory [36, 37]. Consider type IIA on a CY three-fold \mathbb{X} , which provides a physical realization of the topological A-model on \mathbb{X} . The genus g topological string amplitude $F_g(t_i)$, which depends on the complexified Kähler moduli $t_i = a_i + iv_i$ with a_i coming from the B-field and v_i being the volume, computes the F-term [9]

$$\int d^4x \int d^4\theta F_g(t_i) (\mathcal{W}^2)^g \rightarrow \int d^4x F_g(t_i) F_+^{2g-2} R_+^2, \quad (3.2.1)$$

where the second expression applies for $g > 1$ only, and the $\mathcal{N} = 2$ Weyl multiplet is schematically $\mathcal{W} = F_+ + \theta^2 R_+ + \dots$, in terms of the self-dual graviphoton and curvature. If we turn on a constant self-dual graviphoton background in the four non-compact dimensions

$$F_+ = \frac{\epsilon}{2} dx^1 \wedge dx^2 + \frac{\epsilon}{2} dx^3 \wedge dx^4, \quad (3.2.2)$$

the sum may be regarded as the total A-model free-energy, with coupling ϵ

$$\mathcal{F}(t_i) = \sum_{g=0}^{\infty} \epsilon^{2g-2} F_g(t_i). \quad (3.2.3)$$

This same quantity can be directly computed as a one-loop diagram of 5d BPS states in M-theory compactified on $\mathbb{X} \times \mathbb{S}^1$, corresponding to 11d graviton multiplets and to M2-branes wrapped on holomorphic 2-cycles. These states couple to the graviphoton background via their quantum numbers under the $SU(2)_L$ in the 5d little group $SU(2)_L \times SU(2)_R$. Denoting by $GV_{g,\beta}$ the multiplicity of BPS states corresponding to M2-branes on a genus g curve in the homology class β , we have an expression²

$$\mathcal{F} = \sum_{\beta \in H_2(\mathbb{X}; \mathbb{Z})} \sum_{g=0}^{\infty} \sum_{m=1}^{\infty} GV_{g,\beta} \frac{1}{m} \left(2i \sinh \frac{m\epsilon}{2} \right)^{2g-2} e^{im\beta \cdot t}, \quad (3.2.4)$$

with $|e^{im\beta \cdot t}| < 1$ so that the BPS state counting is well-defined. Namely, the computation is carried out in the large volume point in the Kähler moduli space. From the M-theory point of view, the real part of t may be provided by the Wilson line along \mathbb{S}^1 , originated from the three-form C_3 . Finally, the topological string partition function is defined as

$$Z_{\text{top}} = \exp \mathcal{F}. \quad (3.2.5)$$

Topological vertex formalism

From now on we specialize to a particular class of M-theory background geometries, directly related to supersymmetric gauge theories. M-theory on a CY three-fold singularity, in the decoupling limit, implements a geometric engineering realization of 5d gauge theory with 8 supercharges, with gauge group and matter content determined by the singularity structure [50, 71]. Upon compactification on \mathbb{S}^1 , it reproduces type IIA geometric engineering [59, 60]. This setup is therefore well-suited for the matching of gauge theory results in terms of topological string amplitudes in the GV interpretation.

We will use the best known tool at large volume point in moduli space to compute topological string partition functions on local toric CY three-folds, namely the topological vertex formalism [3, 10, 52, 56]. This can be even used to define a refined version of the topological string amplitude [2, 4, 7, 10, 24, 40, 56], associated to the theory in a non self-dual graviphoton background

$$F = \frac{1}{2} \epsilon_1 dx^1 \wedge dx^2 - \frac{1}{2} \epsilon_2 dx^3 \wedge dx^4. \quad (3.2.6)$$

²A subtlety regarding the reality condition of the ϵ -background has been discussed in [25].

The unrefined topological string amplitude is recovered for $\epsilon_1 = -\epsilon_2$. We will describe the refined topological vertex computation, but will eventually restrict to the unrefined case, since only this is known in the unoriented case. Happily, this suffices to illustrate our main points.

The basic idea is to regard the web diagram of the resolved singularity (the dual of the toric fan) as a Feynman diagram, with rules to produce the topological string partition function. Roughly, one sums over edges that correspond to Young diagrams R_i , with propagators $(-e^{it_i})^{|R_i|}$ depending on Kähler parameters t_i , and vertex functions expressed in terms of skew-Schur functions. The formalism is derived from open-closed string duality and 3d Chern-Simons theory [3, 5, 52].

Let us begin by introducing some useful definitions. A Young diagram R is defined by the numbers of boxes $R(i)$ in the i^{th} column, ordered as $R(1) \geq R(2) \geq \dots \geq R(d) \geq R(d+1) = 0$, see fig. 3.1. We denote by $|R| = \sum_{i=1}^d R(i)$ the total number of boxes, and by \emptyset the empty diagram. For a box $s = (i, j) \in R$, we also define

$$a_R(i, j) := R^t(j) - i, \quad l_R(i, j) := R(i) - j, \quad (3.2.7)$$

where R^t denotes transpose, see fig. 3.1.

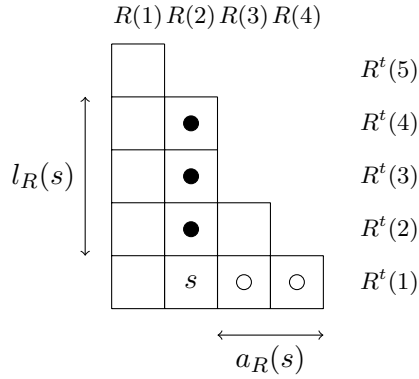


Figure 3.1: Notation for arm and leg lengths of the box $s = (2, 1) \in R$.

We recall some definitions useful to work with refined topological vertex³ by following the conventions used in [44]. An edge is labeled by a Young diagram ν , and has an associated propagator $(-Q_\nu)^{|\nu|}$, where Q is the exponential of the complexified Kähler parameter of the corresponding 2-cycle. Edges join at vertices, which have an associated vertex function

$$C_{\lambda\mu\nu}(t, q) = t^{-\frac{\|\mu^t\|^2}{2}} q^{\frac{\|\mu\|^2 + \|\nu\|^2}{2}} \tilde{Z}_\nu(t, q) \sum_{\eta} \left(\frac{q}{t}\right)^{\frac{|\eta| + |\lambda| - |\mu|}{2}} s_{\lambda^t/\eta}(t^{-\rho} q^{-\nu}) s_{\mu/\eta}(t^{-\nu^t} q^{-\rho}), \quad (3.2.8)$$

where $q = e^{-i\epsilon_2}$ and $t = e^{i\epsilon_1}$, $s_R(q^{-\rho} t^{-\nu})$ means s_R evaluated at $x_i = q^{i-\frac{1}{2}} t^{-\nu(i)}$, and $s_{\mu/\eta}(x)$ are skew-Schur functions: if $s_\nu(x)$ is Schur function and $s_\nu(x) s_\rho(x) =$

³In subindex-packed formulas, we sometimes adopt Greek letters to label Young diagrams.

$\sum_{\mu} c_{\nu\rho}^{\mu} s_{\mu}(x)$, then $s_{\mu/\nu}(x) := \sum_{\rho} c_{\nu\rho}^{\mu} s_{\rho}(x)$ [70] (in particular, $s_{\emptyset/R} = \delta_{\emptyset,R}$, $s_{R/\emptyset} = s_R$, and $s_{\mu/\nu}(Qq) = Q^{|\mu|-|\nu|} s_{\mu/\nu}(q)$.) Sub-indices are ordered according to fig. 3.2a, and we defined

$$\tilde{Z}_{\nu}(t, q) = \prod_{s \in \nu} \left(1 - q^{l_{\nu}(s)} t^{a_{\nu}(s)+1}\right)^{-1}. \quad (3.2.9)$$

We also define

$$f_{\nu}(t, q) = (-1)^{|\nu|} t^{\frac{\|\nu^t\|^2}{2}} q^{-\frac{\|\nu\|^2}{2}}, \quad \tilde{f}_{\nu}(t, q) = (-1)^{|\nu|} t^{\frac{\|\nu^t\|^2}{2}} q^{-\frac{\|\nu\|^2}{2}} \left(\frac{t}{q}\right)^{\frac{|\nu|}{2}}, \quad (3.2.10)$$

so that when we glue two refined topological vertices, we introduce framing factors $\tilde{f}_{\nu^t}(t, q)^n$ or $f_{\nu^t}(q, t)^n$ depending on whether the internal line is the non-preferred direction (fig. 3.2b) or preferred direction (fig. 3.2c) respectively, with $n := \det(u_1, u_2)$. The rule for the unrefined topological vertex is recovered by setting $t = q$.

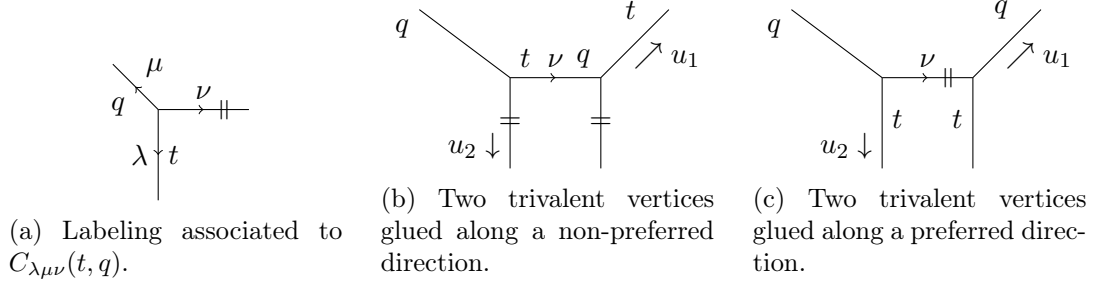


Figure 3.2: Refined vertex conventions. We express a leg in the preferred direction by \parallel .

Examples

SU(N) gauge theory As an illustrative example, consider the toric diagram for an SU(N) gauge theory, given in fig. 3.3, where we use standard notation [3, 53, 54]. Among the different possible ways to get SU(N), we have taken our diagram to be symmetric with respect to a vertical line, for later use in section 3.4 when we impose \mathbb{Z}_2 orientifold involutions. This constrains the slope of external legs in the diagram entering the topological vertex computation (in the gauge theory of the next section, this translates into a choice of 5d Chern-Simons level $K = N - 2$ [83].) We denote by Q_{F_i} the exponential of the Kähler parameter for the edge T_i (for $i = 1, \dots, N - 1$). The exponentials of the Kähler parameters for the horizontal edges R_i are denoted by Q_{B_i} (for $i = 1, \dots, N$), and they can be expressed in terms of $Q_B := Q_{B_1} = Q_{B_2}$ as

$$Q_{B_i} = Q_B \prod_{m=2}^{i-1} Q_{F_m}^{2(m-1)}. \quad (3.2.11)$$

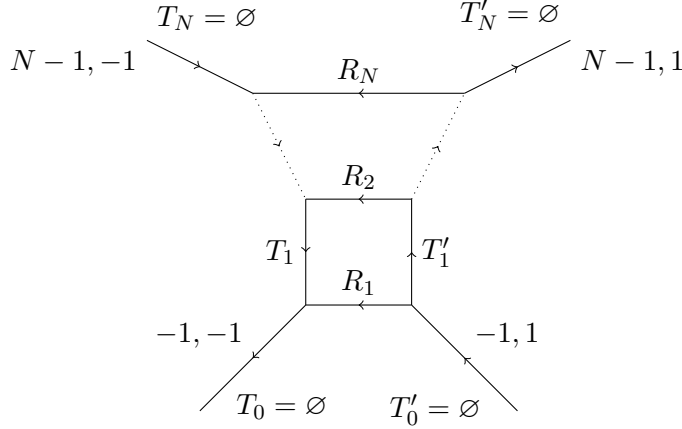


Figure 3.3: Web diagram for a toric singularity engineering pure $SU(N)$ gauge theory with $K = N - 2$.

The refined topological string partition function is written

$$Z_{\text{ref top}}^{\text{SU}(N)} = \sum_{\substack{T_1, \dots, T_{N-1} \\ R_1, \dots, R_N \\ T'_1, \dots, T'_{N-1}}} \prod_{i=1}^N C_{T_{i-1} T_i^t R_i^t}(t, q) \tilde{f}_{T_i^t}(t, q) (-Q_{F_i})^{|T_i|} \times \\ \times f_{R_i}(q, t)^{-2i+3} (-Q_{B_i})^{|R_i|} \times C_{T'_i T'_{i-1} R_i}(q, t) \tilde{f}_{T'_i}(q, t) (-Q_{F_i})^{|T'_i|}, \quad (3.2.12)$$

where the relevant rules and quantities are defined in section 3.2.1. We focus on the unrefined case (i.e. set $t = q = e^{i\epsilon}$), where this can be explicitly evaluated to give

$$Z_{\text{top}}^{\text{SU}(N)} = \sum_{\mathbf{R}} \left(\prod_{i=1}^N q^{(\|R_i^t\|^2 - \|R_i\|^2)(1-i) + \|R_i^t\|^2} \tilde{Z}_{R_i}^2 Q_{B_i}^{|R_i|} \right) \times \\ \times \prod_{1 \leq k < l \leq N} \prod_{i,j=1}^{\infty} \left[1 - \left(\prod_{m=k}^{l-1} Q_{F_m} \right) q^{i+j-1-R_k(i)-R_l^t(j)} \right]^{-2}. \quad (3.2.13)$$

Using eqs. (3.C.1) and (3.C.4) we can recast the expression as sum over N -tuples of Young diagrams $\mathbf{R} = (R_1, \dots, R_N)$ such that $|\mathbf{R}| := \sum_i |R_i| = k$,

$$Z_{\text{top}}^{\text{SU}(N)} = Z_{\text{top pert}}^{\text{SU}(N)} \sum_{k=0}^{\infty} u^k \sum_{|\mathbf{R}|=k} C S_{N,N-2} \prod_{i,j=1}^N \prod_{s \in R_i} \frac{1}{(2i)^2 \sin^2 \frac{1}{2} E_{ij}(s)}, \quad (3.2.14)$$

where

$$E_{ij}(s) = t_i - t_j - \epsilon_1 l_{R_i}(s) + \epsilon_2 [a_{R_j}(s) + 1], \quad (3.2.15)$$

(we also denote by E the unrefined expression, i.e. the one with $\epsilon_1 = -\epsilon_2 =: \epsilon$). Here we have introduced the perturbative contribution

$$Z_{\text{top pert}}^{\text{SU}(N)} := \prod_{1 \leq k < l \leq N} \prod_{i,j=1}^{\infty} \left[1 - \left(\prod_{m=k}^{l-1} Q_{F_m} \right) q^{i+j-1} \right]^{-2} \quad (3.2.16)$$

and the instanton fugacity

$$u := \frac{Q_B}{\left(\prod_{i=1}^{N-1} Q_{F_i}^{N-i} \right)^{\frac{2}{N}}}. \quad (3.2.17)$$

We have also introduced with hindsight the quantities $t_i = a_i + i v_i$, $i = 1, \dots, N$, satisfying $\sum_i t_i = 0$, to rephrase the Kähler parameters as

$$Q_{F_i} = e^{i(t_{i+1} - t_i)}. \quad (3.2.18)$$

$v_{i+1} - v_i$ encodes the length of vertical edge F_i in the web diagram (and eventually the gauge theory Coulomb branch parameters). They should satisfy $|Q_{F_i}| < 1$ or $v_{i+1} > v_i$, so that it is a good expansion parameter. In other words, we are at a large volume point in the Kähler moduli space spanned by v_i . We also have the quantity (eventually corresponding to the contribution from the gauge theory Chern-Simons term)

$$CS_{N,K} := \prod_{i=1}^N \prod_{s \in R_i} e^{i K E_{i\partial}(s)}. \quad (3.2.19)$$

Conifold Another illustrative example is the resolved conifold, whose web diagram (again restricting to a case symmetric under a line reflection, for future use) is given in fig. 3.4. The unrefined topological vertex computation gives

$$Z_{\text{top}}^{\text{U}(1)} = \exp \left\{ - \sum_{m=1}^{\infty} \frac{1}{m} \frac{Q^m}{(q^{m/2} - q^{-m/2})^2} \right\} = \prod_{n=1}^{\infty} (1 - Q q^n)^n, \quad (3.2.20)$$

where Q is the Kähler parameter.

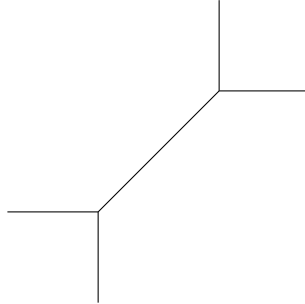


Figure 3.4: Web diagram for the conifold.

3.2.2. Supersymmetric Yang-Mills on ϵ -background

We now consider a 4d $\mathcal{N} = 2$ gauge theory. Its exact quantum dynamics is obtained by the perturbative one-loop contribution and the contribution from the (infinite) set of BPS instantons. These corrections can be obtained from a 5d theory with 8 supercharges, compactified on \mathbb{S}^1 , as a one-loop contribution from the set of 5d one-particle BPS states. These particles are perturbative states of the 5d theory and BPS instanton particles.

The 5d instanton partition function [76] is given by a power series expansion $Z_{\text{inst}} = \sum_{k=0}^{\infty} u^k Z_k^{\text{inst}}$, where the contribution for instanton number $k = c_2$ is

$$Z_k^{\text{inst}}(\{a_i\}; \epsilon_1, \epsilon_2) = \text{tr}_{\mathcal{H}_k} \left[(-1)^F e^{-\beta H} e^{-i\epsilon_1(J_1 + J_{\mathcal{R}})} e^{-i\epsilon_2(J_2 + J_{\mathcal{R}})} e^{-i \sum a_i \Pi_i} \right]. \quad (3.2.21)$$

Here we trace over the 5d Hilbert space \mathcal{H} of one-particle massive BPS states. Also, J_1 and J_2 span the Cartan subalgebra of the $\text{SO}(4)$ little group, $J_{\mathcal{R}}$ the Cartan of the $\text{SU}(2)_{\mathcal{R}}$ R-symmetry, and Π are the Cartan generators for a gauge group G . a_i , $i = 1, \dots, \text{rank } G$ are Wilson lines of G on the \mathbb{S}^1 .

The BPS particles are W-bosons, 4d instantons (viewed as solitons in the 5d theory) and bound states thereof. For $k \neq 0$, the partition function can be regarded as a Witten index in the SUSY quantum mechanics whose vacuum is the ADHM moduli space, with the SUSY algebra

$$\{Q_M^A, Q_N^B\} = P_\mu (\Gamma^\mu C)_{MN} \epsilon^{AB} + i \frac{4\pi^2 k}{g_{\text{YM}}^2} C_{MN} \epsilon^{AB} + i \text{tr}(v \Pi) C_{MN} \epsilon^{AB}, \quad (3.2.22)$$

where v_i are the 5d Coulomb branch parameters. We will eventually complexify them with the already appeared Wilson lines to complete the complex Coulomb branch parameters, bearing in mind that we will compare the Nekrasov partition function in eq. (3.2.21) with the topological string partition function in eq. (3.2.5).

In the following we focus on $G = \text{U}(N)$ with 5d Chern-Simons level K . The quantity in eq. (3.2.21) can be evaluated by using localization in equivariant K-theory [73, 74, 76] on the instanton moduli space $M(N, k)$, more precisely on its Gieseker partial compactification and desingularization given by framed rank N torsion-free sheaves on $\mathbb{CP}^2 = \mathbb{R}^4 \cup \ell_\infty$, where the framing is given by a choice of trivialization on the line at infinity ℓ_∞ . The ϵ -background localizes the integral, restricting it to a sum over fixed points of the equivariant action.

The result of computation for pure $G = \text{U}(N)$ gauge theory with Chern-Simons level K is [23, 49, 74, 76, 83, 84]

$$Z_{\text{inst}}^{\text{U}(N)} = \sum_{k=0}^{\infty} u^k \sum_{|\mathbf{R}|=k} \prod_{i=1}^N \prod_{s \in R_i} \frac{e^{iK(E_{i\emptyset}(s) - \frac{1}{2}(\epsilon_1 + \epsilon_2))}}{\prod_{j=1}^N (2i)^2 \sin \frac{E_{ij}(s)}{2} \sin \frac{E_{ij}(s) - (\epsilon_1 + \epsilon_2)}{2}}. \quad (3.2.23)$$

For comparison with the topological string result, we take $K = N - 2$. We can restrict the result to $\text{SU}(N)$ by constraining the sum of the Coulomb branch moduli to be zero. Note

that the $U(N)$ and $SU(N)$ instanton partition functions are in general different,⁴ but they agree in our case of zero flavors with Chern-Simons level $K = N - 2$. With this proviso, we can see that eq. (3.2.14) from section 3.2.1 can be written as $Z_{\text{top}}^{\text{SU}(N)} / Z_{\text{top pert}}^{\text{SU}(N)} = Z_{\text{inst}}^{\text{U}(N)}$ evaluated for $K = N - 2$, $\sum t_i = 0$, and in the unrefined limit $\epsilon_1 = -\epsilon_2 =: \epsilon$, by identifying the complexified Kähler parameter with the complexified Coulomb branch moduli. Hence we have an exact match up to the perturbative part.

Another interesting example is $U(1)$ gauge theory. Although it does not support semiclassical gauge instantons, one can consider BPS states corresponding to small instantons. A mathematically more rigorous way to define them is to consider $U(1)$ instantons on non-commutative \mathbb{R}^4 , or equivalently rank 1 torsion-free sheaves on \mathbb{CP}^2 with fixed framing on the line at infinity [75]. The gauge theory result is [72, 82]

$$\sum_R \frac{Q^{|R|}}{\prod_{s \in R} \left(1 - q_1^{-l_R(s)} q_2^{1+a_R(s)}\right) \left(1 - q_1^{1+l_R(s)} q_2^{-a_R(s)}\right)} = \exp \left\{ \sum_{r=1}^{\infty} \frac{1}{r} \frac{Q^r}{(1 - q_1^r)(1 - q_2^r)} \right\}, \quad (3.2.24)$$

where $q_1 = e^{i\epsilon_1}$, $q_2 = e^{i\epsilon_2}$. The exponent agrees precisely with the topological vertex result eq. (3.2.20) in the unrefined case $q_1 q_2 = 1$, by setting $q_1 = q$.

3.3. Orientifolds and the real topological vertex

In this section we review properties of the unoriented theories we are going to focus on. We first introduce their description in string theory and M-theory, and subsequently review the computation of their partition function using the real topological string theory in the real topological vertex formalism.

3.3.1. Generalities

There are many ways to obtain an unoriented theory from a parent oriented string theory configuration, which in our present setup result in different gauge theory configurations. We will focus on a particular choice, which has the cleanest connection with the parent oriented theory, in a sense that we now explain.

Consider the type IIA version of our systems, namely type IIA on a non-compact toric CY threefold \mathbb{X} singularity. We introduce an orientifold quotient, acting as an antiholomorphic involution σ on \mathbb{X} and as a sign flip in an \mathbb{R}^2 (parametrized by x^2, x^3) of 4d Minkowski space. For concreteness, we consider σ to have a fixed locus L , which on general grounds is a lagrangian 3-cycle of \mathbb{X} (one can build orientifolds with similar M-theory lift even if σ is freely acting). In other words, we have an O4-plane wrapped on L and spanning x^0, x^1 ; we choose the O4-plane to carry negative RR charge (see later for other choices). We complete the configuration by introducing one single D4-brane (as

⁴The difference between the $U(N)$ Nekrasov partition functions and the $SU(N)$ Nekrasov partition functions has been discussed in [12, 14, 15, 44, 49]. It turns out that the web diagram nicely encodes factors that account for the difference.

counted in the covering space) wrapped on L and spanning x^0, x^1 , namely stuck on the $O4^-$ -plane.⁵

In general, if $H_1(L; \mathbb{Z})$ is non-trivial, it is possible to turn on \mathbb{Z}_2 -valued Wilson lines for the D4-branes worldvolume $O(1) \equiv \mathbb{Z}_2$ gauge group. This results in a different sign weight for the corresponding disk amplitudes, as discussed in the explicit example later on.

This setup is the physical realization of the real topological string introduced in [91] (see also [65–67]). In the topological setup, the addition of the D4-brane corresponds to a topological tadpole cancellation condition; in the physical setup, it corresponds to local cancellation of the RR charge, and leads to a remarkably simple M-theory lift, which allows for direct connection with the 5d picture of the oriented case, as follows.

This IIA configuration lifts to M-theory as a compactification on $\mathbb{X} \times \mathbb{S}^1$, with a \mathbb{Z}_2 quotient⁶ acting as σ on \mathbb{X} , as $(x^2, x^3) \rightarrow (-x^2, -x^3)$ on 4d Minkowski space-time, and as a half-shift along the \mathbb{S}^1 . Because the \mathbb{Z}_2 is freely acting on the \mathbb{S}^1 , the configuration can be regarded as an \mathbb{S}^1 compactification of the 5d theory corresponding to M-theory on \mathbb{X} (with the \mathbb{S}^1 boundary conditions for the different fields given by their eigenvalue under the orientifold action). Since the 5d picture is essentially as in the oriented case, these configurations have a direct relation with the oriented Gopakumar-Vafa description of the topological string. Specifically, the real topological string amplitude is given by a one-loop diagram of 5d BPS states running on \mathbb{S}^1 , with integer (resp. half integer) KK momentum for states even (resp. odd) under the orientifold action [79].

For completeness, we quote the M-theory lifts corresponding to other choices of $O4^-$ plane and D4-brane configurations [35, 47]:

- An orientifold introducing an $O4^-$ -plane with no stuck D4-brane lifts to M-theory on $\mathbb{X} \times \mathbb{S}^1$ with a \mathbb{Z}_2 acting as σ on \mathbb{X} , flipping x^2, x^3 in 4d space, and leaving the \mathbb{S}^1 invariant. This M-theory configuration has orbifold fixed points and therefore is not directly related to the 5d picture of the oriented theory.
- An orientifold introducing an $O4^+$ -plane lifts to M-theory on $\mathbb{X} \times \mathbb{S}^1$ with a \mathbb{Z}_2 acting as σ on \mathbb{X} , flipping x^2, x^3 in 4d space, and leaving the \mathbb{S}^1 invariant, with 2 M5-branes stuck at the orbifold locus. Again this M-theory configuration has orbifold fixed points.
- Finally, there is an exotic orientifold, denoted $\widetilde{O4^+}$ -plane, which lifts to M-theory as our above freely acting orbifold (acting with a half-shift on \mathbb{S}^1), with one extra stuck M5-brane. This M-theory configuration contains a sector of closed membranes exactly as in the $O4^-+D4$ case, and in addition an open membrane sector which has no direct relation to the 5d oriented theory (but is described by Ooguri-Vafa invariants [77]).

⁵Additional pairs of mirror D4-branes may be added; in the M-theory lift, they correspond to the inclusion of explicit M5-branes, so that open M2-brane states enter the computations.

⁶The M-theory 3-form C_3 is intrinsically odd under this \mathbb{Z}_2 , so we refer to it as ‘orientifold action’ in M-theory as well.

Hence, as anticipated, we focus on the $O4^- + D4$, whose M-theory lift is the simplest and closest to the parent 5d oriented theory.

To finally determine the orientifold actions, we must specify the antiholomorphic involution σ acting on \mathbb{X} . In general, a toric CY three-fold associated to an $SU(N)$ gauge theory admits two such \mathbb{Z}_2 actions,⁷ illustrated in fig. 3.5 for $SU(4)$. They mainly differ in the effect of the orientifold action on the Coulomb branch moduli of the 5d gauge theory. Namely, the blue quotient in fig. 3.5 reduces the number of independent moduli, whereas the red one preserves this number. Equivalently, the two quotients either reduce or preserve the rank of the gauge group at the orientifold fixed locus. Since the Coulomb branch parameters play an important role in the parent gauge theory localization computation, we will focus on rank-preserving quotients to keep the discussion close to the parent theories. We leave the discussion of rank-reducing involutions for future projects.

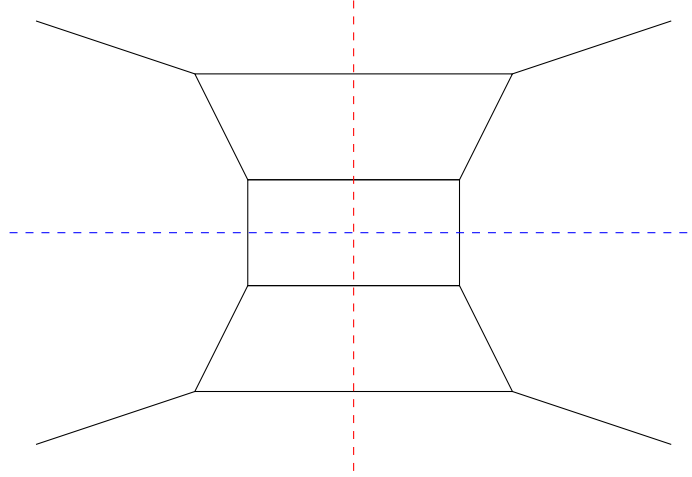


Figure 3.5: $SU(4)$ with two involutions.

3.3.2. The real topological string

The real topological string is a natural generalization of the topological string in section 3.2.1. It provides a topological version of the IIA orientifolds in the previous section. Namely, the real topological string computes holomorphic maps from surfaces with boundaries and crosscaps into a target \mathbb{X} modded out by the orientifold involution σ . Realizing the unoriented world-sheet surface as a quotient of a Riemann surface by an antiholomorphic involution,⁸ $\Sigma = \Sigma_g / \Omega$, we must consider equivariant maps f as in fig. 3.6.

⁷Certain cases, like the conifold, may admit additional symmetries.

⁸The case Σ itself is a Riemann surface requires to start from a disconnected Σ_g , and is better treated separately.

$$\begin{array}{ccc}
\Sigma_g & \xrightarrow{\Omega} & \Sigma_g \\
f \downarrow & \circlearrowleft & \downarrow f \\
\mathbb{X} & \xrightarrow{\sigma} & \mathbb{X}
\end{array}$$

Figure 3.6: Commutative diagram for equivariant map.

The model includes crosscaps, and boundaries (with a single-valued Chan-Paton index to achieve the topological tadpole cancellation) ending on the lagrangian L . Hence, we must consider the relative homology class $f_*([\Sigma_g]) \in H_2(\mathbb{X}, L; \mathbb{Z})$.

The total topological amplitude at fixed Euler characteristic χ may be written as

$$\mathcal{G}^{(\chi)} = \frac{1}{2} \left[\mathcal{F}^{(g_\chi)} + \sum \mathcal{F}^{(g,h)} + \sum \mathcal{R}^{(g,h)} + \sum \mathcal{K}^{(g,h)} \right], \quad (3.3.1)$$

where the different terms account for closed oriented surfaces, oriented surfaces with boundaries, surfaces with one crosscap, and surfaces with two crosscaps. Different consistency conditions, needed to cancel otherwise ill-defined contributions from the enumerative geometry viewpoint,⁹ guarantee integrality of the BPS expansion for

$$\mathcal{F}_{\text{real}} = \sum_{\chi} i^{\chi} \epsilon^{\chi} \left(\mathcal{G}^{(\chi)} - \frac{1}{2} \mathcal{F}^{(g_\chi)} \right). \quad (3.3.2)$$

This can be taken as the definition of the real topological string.

This integrality of BPS invariants, as well as a physical explanation of the tadpole cancellation and other consistency conditions of the real topological string, may be derived from the M-theory viewpoint [79]. The real topological string amplitude is obtained as a sum over 5d BPS M2-brane states of the oriented theory, running in the compactification \mathbb{S}^1 with boundary conditions determined by the eigenvalue under the orientifold operator. For a short review, see section 3.A.1. Denoting by $\widehat{\text{GV}}_{\hat{g},\beta}$ this weighted BPS multiplicity of M2-branes wrapped on a genus \hat{g} surface (as counted in the quotient) in the homology class β , the equivalent to eq. (3.2.4) is

$$\mathcal{F}_{\text{real}} = \sum_{\substack{\beta, \hat{g} \\ \text{odd } m \geq 1}} \widehat{\text{GV}}_{\hat{g},\beta} \frac{1}{m} \left[2 \sinh \left(\frac{m\epsilon}{2} \right) \right]^{\hat{g}-1} e^{im\beta \cdot t}. \quad (3.3.3)$$

To compute real topological string partition function on local Calabi-Yau, we will use the real topological vertex [65], which is a generalization of the standard topological vertex to take into account involutions of the toric diagram. The formalism is still only available in the unrefined case, on which we focus herefrom.

We apply the formalism to involutions of the kind shown in fig. 3.7, as described in more detail in section 3.A.2. For these involutions there are no legs fixed point-wise in

⁹For example, for configurations in which $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$ and $H_2(\mathbb{X}, L; \mathbb{Z}) = \mathbb{Z}$, one needs to cancel homologically trivial disks against crosscaps.

the diagram, and this simplifies the computation of the topological vertex. Due to the symmetry of the diagram in the parent theory, one can use symmetry properties of the vertex functions to cast each summand in the sum over Young diagrams as a square [65]. Then the real topological vertex amplitude is given by the sum of the square roots of the summands. To define these in a consistent way, we follow the choice of sign in [65], see eq. (3.A.6). In all our examples this sign is trivial, since $|R| \pm c(R)$ is even for every R , and in the cases we consider also $n + 1 = -2i + 4$ is even as well, as can be seen from eq. (3.2.12).

Explicit examples will be described in section 3.4.

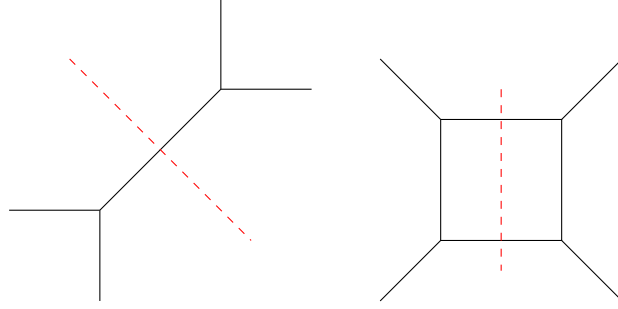


Figure 3.7: Web diagram for our orientifolds of the conifold and $SU(2)$ theories.

3.4. Explicit examples

In this section, we explicitly compute the real topological string partition functions of the resolved conifold and also the $SU(N)$ geometry using the real topological vertex formalism [65]. We first review the calculation of the real topological string partition function of the resolved conifold [65], correcting some typos. Then, we move on to the computation of the real topological string partition function for the $SU(N)$ geometry, which shows an intriguing new feature.

3.4.1. The real conifold

Let us apply the above recipe to the orientifold of the resolved conifold. This is particularly simple because there are no Coulomb branch moduli, and the only parameters are the instanton fugacity and those defining the ϵ -background.

The topological string side can be computed using the real topological vertex formalism. The result¹⁰ reads

$$Z_{\text{real top}}^{U(1)} = \exp \left\{ -\frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \frac{Q^m}{(q^{m/2} - q^{-m/2})^2} \pm \sum_{m \text{ odd}} \frac{1}{m} \frac{Q^{m/2}}{q^{m/2} - q^{-m/2}} \right\}. \quad (3.4.1)$$

¹⁰This expression corrects some typos in [65].

Our choice of orientifold plane charge corresponds to the negative sign.¹¹

The first term in the exponent corresponds to the closed topological string contribution, while the second reproduces the open and unoriented topological string contributions.

3.4.2. Orientifold of pure $SU(N)$ geometry, and its flops

In this section we study the unoriented version of the $SU(N)$ systems of section 3.2.1, compute their real topological vertex amplitudes following the rules in [65], and describe their behavior under flops of the geometry.

Real topological vertex computation

The web diagram is given in fig. 3.8, which describes a \mathbb{Z}_2 involution of fig. 3.3 (which was chosen symmetric in hindsight).

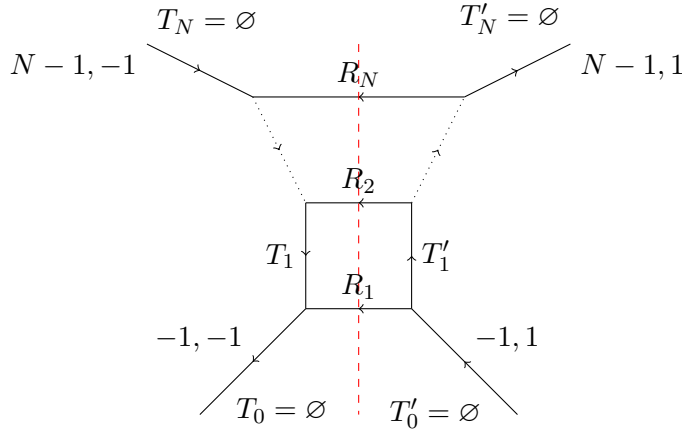


Figure 3.8: $SU(N)$ with $K = N - 2$ and involution.

We recall some expressions already introduced in section 3.2.1 for the oriented case. We define the perturbative contribution as

$$Z_{\text{real top pert}}^{SU(N)} := \prod_{1 \leq k < l \leq N} \prod_{i,j=1}^{\infty} \left(1 - \left(\prod_{m=k}^{l-1} Q_{F_m} \right) q^{i+j-1} \right)^{-1}. \quad (3.4.2)$$

We also recall the Chern-Simons level eq. (3.2.19)

$$CS_{N,K} := \prod_{i=1}^N \prod_{s \in R_i} e^{iK E_{i\emptyset}(s)}, \quad (3.4.3)$$

¹¹The choice of positive sign can be recovered by turning on a non-trivial \mathbb{Z}_2 -valued Wilson line on the D4-brane stuck at the O4-plane in the type IIA picture, since the fixed locus $L = \mathbb{R}^2 \times \mathbb{S}^1$ has one non-trivial circle.

where E_{ij} is defined in eq. (3.2.15) and \emptyset denotes the empty diagram. Finally, we introduce the rescaled instanton fugacity

$$\tilde{u} := \frac{Q_B^{\frac{1}{2}}}{\left(\prod_{i=1}^{N-1} Q_{F_i}^{N-i}\right)^{\frac{1}{N}}}, \quad (3.4.4)$$

which is the square root of the instanton fugacity eq. (3.2.17) in the oriented computation.

Expressing the (complexified) Kähler parameters in terms of the (complexified) edge positions t_i , with $\sum_i t_i = 0$, and edges ordered such that $\text{Im } t_{i+1} > \text{Im } t_i$ in a certain large volume region in the Kähler moduli space, we take as in eq. (3.2.18)

$$Q_{F_i} = e^{i(t_{i+1}-t_i)}. \quad (3.4.5)$$

Notice that in this case the fugacity can be written as $\tilde{u} = Q_B^{\frac{1}{2}} e^{it_1}$.

The computation is as follows: we start from eq. (3.2.12), go to the unrefined limit, and apply real topological vertex rules [65], cf. section 3.A.2. Since our involution does not fix any leg point-wise, we only need to reconstruct the square within summands using permutation properties of the topological vertex $C_{RR'R''}(q)$ (see eq. (3.A.4)), and the fact that $T_i = T'_i$ due to the involution. We get

$$Z_{\text{top}}^{\text{SU}(N)} \Big|_{T_i=T'_i} = \sum_{\substack{T_1, \dots, T_{N-1} \\ R_1, \dots, R_N}} \prod_{i=1}^N C_{T_i T_{i-1}^t R_i}(q) Q_{F_i}^{2|T_i|} Q_{B_i}^{|R_i|} (-1)^{|R_i|(4-2i)} q^{\|T_i\|^2 - \|T_i^t\|^2 + (2-i)(\|R_i^t\|^2 - \|R_i\|^2)}. \quad (3.4.6)$$

We then take the square root, and notice that the sign eq. (3.A.6) for the propagator is always +1,

$$Z_{\text{real top}}^{\text{SU}(N)} = \sum_{\substack{T_1, \dots, T_{N-1} \\ R_1, \dots, R_N}} \prod_{i=1}^N C_{T_i T_{i-1}^t R_i}(q) Q_{F_i}^{|T_i|} Q_{B_i}^{|R_i|/2} q^{\frac{1}{4}(\|T_i\|^2 - \|T_i^t\|^2) + \frac{1}{4}(\|T_{i-1}\|^2 - \|T_{i-1}^t\|^2) + \frac{2-i}{2}(\|R_i^t\|^2 - \|R_i\|^2)}. \quad (3.4.7)$$

By using combinatorial identities for Young diagrams and skew-Schur functions, described in section 3.C (in particular eqs. (3.C.1) and (3.C.5)), we arrive at the final result

$$Z_{\text{real top}}^{\text{SU}(N)} = Z_{\text{real top pert}}^{\text{SU}(N)} \sum_{k=0}^{\infty} \tilde{u}^k \sum_{|\mathbf{R}|=k} (-1)^{\sum_i (i-1)|R_i|} CS_{N, \frac{N-2}{2}} \prod_{i,j=1}^N \prod_{s \in R_i} \frac{1}{2i \sin \frac{1}{2} E_{ij}(s)}. \quad (3.4.8)$$

Behavior under flops

It is worthwhile to briefly step back and emphasize an important point. In the above computation there is an explicit choice of ordering of edges in the web diagram, which defines a particular large volume limit. Moving in the Kähler moduli space across a wall of a flop transition¹² can reorder the edges, so we need to redefine the expansion parameters eq. (3.2.18). Consider the simplest setup in which the ordering of all edges is reversed, such that $v_i > v_{i+1}$ for all i , and we take

$$Q_{F_i} = e^{i(t_{N-i} - t_{N-i+1})}, \quad (3.4.9)$$

with $|Q_{F_i}| < 1$. In this case, we are at a different large volume point in the enlarged Kähler moduli space compared to the case when we defined t_i by eq. (3.4.5). From the viewpoint of the five-dimensional pure $SU(N)$ gauge theory, it corresponds to moving to a different Weyl chamber in the Coulomb branch moduli space by a Weyl transformation $t_i \rightarrow t_{N-i+1}$ for all i . If we write the result by using $E_{ij}(s)$ and $CS_{N,K}$ defined in eqs. (3.2.15) and (3.2.19), this gives the same exact result *except* for sign pattern $(-1)^{\sum_i (N-i)|R_i|}$. The computation of this result is similar to eq. (3.4.8) except that we redefined dummy variables $R_i^{\text{new}} := (R_{N-i+1}^{\text{old}})^t$, compared to the geometry before the flop transition. When we regard eq. (3.4.8) as a function of t_i , we have computed

$$Z_{\text{real top}}(t_{N-i+1}, K) =: \tilde{Z}_{\text{real top}}(t_i, K), \quad (3.4.10)$$

which is not equal to $Z_{\text{real top}}(t_i, K)$ as a function of t_i in the unoriented case.¹³ This is different from the oriented case where we have $Z_{\text{top}}(t_i, K) = Z_{\text{top}}(t_{N-i+1}, K)$, which should be true since Weyl transformations are part of the gauge transformations. Therefore, this is a feature special to the real topological string partition function of pure $SU(N)$ geometry. From the viewpoint of five-dimensional pure $SU(N)$ gauge theory, the pure $SU(N)$ gauge theory is invariant under the Weyl transformation of $SU(N)$ and this is reflected into the invariance of the partition function under the Weyl transformation in the oriented case. In the unoriented case, however, the non-invariance of the partition function under the transformation implies that the presence of the orientifold or the corresponding defect in field theory breaks the symmetry that existed in the oriented case.

It is similarly easy to consider intermediate cases of partial reorderings. The simplest is to take $N = 3$, and move from $v_1 < v_2 < v_3$ to $v_2 < v_3 < v_1$. In this case, the new expansion parameters are $Q_{F_1} = e^{i(t_3 - t_2)}$ and $Q_{F_2} = e^{i(t_1 - t_3)}$. The result is basically the same, but with sign $(-1)^{2|R_1| + |R_3|}$. Here we redefined dummy variables as $R_1^{\text{new}} := (R_3^{\text{old}})^t$, $R_2^{\text{new}} := (R_1^{\text{old}})^t$, $R_3^{\text{new}} := (R_2^{\text{old}})^t$.

In other words, starting with the result in a given chamber, moving across a wall of a flop transition exchanging two edges with diagrams R, S produces a change in the amplitude (expressed in the new Kähler parameters) given by a sign $(-1)^{|R| + |S|}$.

¹²The flop transition considered here shrinks and introduces a family of rational curves. This is different from the usual flop which shrinks and introduces an isolated rational curve. The behavior of the (non-real) topological string partition function under the usual flop has been studied in [55, 63, 85].

¹³In the case $N = 3$ they are equal to each other accidentally.

This is the explicit manifestation of the fact that the topological string amplitude regarded as a function of t_i in this unoriented theory is not universal throughout the moduli space, but it has a non-trivial behavior.¹⁴

Behavior under other transformations

Let us consider another transformation which is a reflection with respect to a horizontal axis for the pure $SU(N)$ geometry of fig. 3.8. The operation in the original pure $SU(N)$ geometry corresponds to charge conjugation, which is given by a transformation of the Coulomb branch moduli $t_i \rightarrow -t_{N-i+1}$ for $i = 1, \dots, N$ and a flip of the sign of CS level.¹⁵ The transformation can be effectively implemented by defining the Kähler parameters as

$$Q_{F_i} = e^{i(t_i - t_{i+1})}, \quad (3.4.11)$$

with $|Q_{F_i}| < 1$ and the same labeling for R_i for all i as in fig. 3.8. Compared to eq. (3.4.5), we flip the sign of t_i for all i . Hence, we are now assuming $v_i > v_{i+1}$ for all i and hence we effectively consider the pure $SU(N)$ geometry upside-down. If we write the result by using $E_{ij}(s)$ and $CS_{N,K}$ defined in eqs. (3.2.15) and (3.2.19), this gives the same result except for different CS level $-\frac{N-2}{2}$ and sign pattern $(-1)^{\sum_i (N-i)|R_i|}$. In this case, we did not redefine the dummy variables R_i . The change in the sign of CS level is consistent with the fact that the definition eq. (3.4.11) is related to charge conjugation of the original pure $SU(N)$ gauge theory. When we regard eq. (3.4.8) as a function of t_i , we have computed

$$Z_{\text{real top}}(-t_i, -K) =: \tilde{Z}_{\text{real top}}(t_i, K), \quad (3.4.12)$$

which is again not equal to $Z_{\text{real top}}(t_i, K)$ as a function of t_i in the unoriented case. This is different from the oriented case where we have $Z_{\text{top}}(t_i, K) = Z_{\text{top}}(-t_i, -K)$.¹⁶ This is another example of a transformation where the presence of the orientifold defect breaks the invariance of the partition function under a transformation that existed in the original theory without the defect.

3.5. Discussion: towards gauge theory interpretation

As explained in the introduction, the real topological string on the local CY threefold should be related to the partition function of the corresponding gauge theory in the presence of a surface defect. Given the M-theory lift of the orientifold in terms of a freely-acting shift on the \mathbb{S}^1 , this should correspond to a partition function of the theory

¹⁴However, when we regard the topological string amplitude as a function of a good expansion parameter which is always Q_{F_i} with $|Q_{F_i}| < 1$, then they are essentially the same function. Namely, the real GV invariants are the same at the two different points in the enlarged Kähler moduli space.

¹⁵When we regard the pure $SU(N)$ geometry as a 5-brane web diagram, the CS level can be read off from the asymptotic behavior of the external legs [13, 14, 62]. In particular, when we turn the 5-brane web upside-down, the sign of the CS level of the gauge theory also flips.

¹⁶We have checked this for $N = 2, 3, 4$ up to 5-instanton order.

on \mathbb{S}^1 , with modified boundary conditions, or equivalently with an extra twist in the Witten index computation,

$$Z_k^{\text{real inst}}(\{a_i\}; \epsilon_1, \epsilon_2) = \text{tr}_{\mathcal{H}_k} \left[(-1)^F e^{-\beta H} e^{-i\epsilon_1(J_1 + J_{\mathcal{R}})} e^{-i\epsilon_2(J_2 + J_{\mathcal{R}})} e^{-i \sum a_i \Pi_i} \mathcal{O}_{\Omega} \right], \quad (3.5.1)$$

where \mathcal{O}_{Ω} is an operator implementing the orientifold action in the corresponding Hilbert space sector.

In this section we exploit the intuitions from the topological vertex computations to describe aspects of this twist in explicit examples.

3.5.1. Invariant states and the conifold example

We start the discussion with the conifold. This is particularly simple, because there is only one BPS (half-)hypermultiplet, whose internal structure is invariant under the orientifold, namely it is an M2-brane wrapped on a 2-cycle mapped to itself under the orientifold. Then the orientifold action is just action on the Lorentz quantum numbers. From the viewpoint of gauge theory, there is a 5d U(1) gauge theory, whose BPS states are instantons. In this simple system it is possible to motivate the structure of the twisted Nekrasov partition function eq. (3.5.1), i.e. of the operator \mathcal{O}_{Ω} . As discussed in section 3.2.1, the ADHM moduli space is just $(\mathbb{C}^2)^k / S_k$ where k is the instanton number and S_k is the symmetric group of order k ; these moduli are intuitively the positions of the k instantons in \mathbb{R}^4 . Therefore the action of \mathcal{O}_{Ω} on this moduli space is simply the geometric action imposed by the orientifold. Since the orientifold action flips the space-time coordinates $(x^2, x^3) \rightarrow (-x^2, -x^3)$, we are motivated to take \mathcal{O}_{Ω} as given by a shift

$$\epsilon_2 \rightarrow \epsilon_2 + \pi \quad (3.5.2)$$

in the original parent gauge theory expression eq. (3.2.21). Furthermore, we assume that the operator induces the redefinition by a factor of 2 of certain quantities between the parent theory and the twisted theory. In practice, it requires that the twisted theory result should be expressed in terms of the redefined weight

$$Q \rightarrow Q^{\frac{1}{2}}. \quad (3.5.3)$$

We then consider the twisted Nekrasov partition function of the U(1) instanton. First, we consider the refined amplitude for the original theory eq. (3.2.24), and perform the shift $\epsilon_2 \rightarrow \epsilon_2 + \pi$. Taking the unrefined limit, we obtain

$$\exp \left\{ -\frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \frac{Q^{2m}}{(2i \sin \epsilon m)^2} - \sum_{k \text{ odd}} \frac{1}{k} \frac{Q^k}{2i \sin \epsilon k} \right\}. \quad (3.5.4)$$

We now redefine $\epsilon = \tilde{\epsilon}/2$ and $Q = \tilde{Q}^{\frac{1}{2}}$, and get

$$Z_{\text{real inst}}^{\text{U(1)}} = \exp \left\{ -\frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \frac{\tilde{Q}^m}{(2i \sin \frac{1}{2} \tilde{\epsilon} m)^2} - \sum_{k \text{ odd}} \frac{1}{k} \frac{\tilde{Q}^{k/2}}{2i \sin \frac{1}{2} \tilde{\epsilon} k} \right\}, \quad (3.5.5)$$

which agrees with eq. (3.4.1) for the negative overall sign for the unoriented contribution. One may choose a redefinition $Q = -\tilde{Q}^{\frac{1}{2}}$, which agrees with eq. (3.4.1) for the positive overall sign for the unoriented contribution. This choice reflects the choice of \mathbb{Z}_2 Wilson line, although its gauge theory interpretation is unclear.

Note that the conifold geometry is special in that the only degree of freedom is one BPS (half-)hypermultiplet, from an M2-brane on a \mathbb{CP}^1 invariant under the orientifold action, which thus motivates a very simple proposal for \mathcal{O}_Ω . This can in general change in more involved geometries, where there are higher spin states, and/or states not invariant under the orientifold. In these cases, the orientifold action should contain additional information beyond its action on 4d space-time quantum numbers.

3.5.2. A twisted Nekrasov partition function for pure $SU(N)$

We now consider the case of the $SU(N)$ geometry, whose real topological string amplitude was described in section 3.4.2, and discuss its interpretation in terms of a twisted Nekrasov partition function.

We start with the following observation. Consider the parent theory expression eq. (3.2.23) for CS level $K = N - 2$ as starting point. Since part of the orientifold action includes a space-time rotation which motivates the ϵ_2 -shift, let us carry it out just like in the conifold case and check the result.

Let us thus perform the ϵ_2 -shift and take the unrefined limit ($\epsilon_2 \rightarrow -\epsilon + \pi$, $\epsilon_1 \rightarrow \epsilon$), and rescale $\epsilon \rightarrow \frac{\epsilon}{2}$, and $t_i \rightarrow \frac{t_i}{2}$ with $u \rightarrow \tilde{u}(= u^{\frac{1}{2}})$. We obtain the result

$$Z_{\text{real inst}}^{\text{SU}(N)} = \sum_{k=0}^{\infty} \tilde{u}^k \sum_{|\mathbf{R}|=k} (-1)^{k+A+\Gamma} \prod_{i=1}^N \prod_{s \in R_i} \frac{e^{i \frac{N-2}{2} E_{i\emptyset}(s)}}{\prod_{j=1}^N (2i) \sin \frac{1}{2} E_{ij}(s)}, \quad (3.5.6)$$

where $A = \frac{1}{2}N (\sum_i \|R_i^t\|^2 + |R_i|)$, and $\Gamma = \sum_{ij} \sum_{s \in R_i} a_{R_j}(s)$. One can prove that $A + \Gamma = \sum_{ij} \sum_{(m,n) \in R_i} R_j^t(n) \equiv k \pmod{2}$, so the final result is

$$Z_{\text{real inst}}^{\text{SU}(N)} = \sum_{k=0}^{\infty} \tilde{u}^k \sum_{|\mathbf{R}|=k} \prod_{i=1}^N \prod_{s \in R_i} \frac{e^{i \frac{N-2}{2} E_{i\emptyset}(s)}}{\prod_{j=1}^N (2i) \sin \frac{1}{2} E_{ij}(s)}. \quad (3.5.7)$$

This expression evaluated for $\sum t_i = 0$ is remarkably close to the real topological string computation eq. (3.4.8), up to i -dependent sign factors. Specifically eq. (3.4.8) can be recast as

$$Z_{\text{real top}}^{\text{SU}(N)} / Z_{\text{real top pert}}^{\text{SU}(N)} = \sum_{k=0}^{\infty} \tilde{u}^k \sum_{|\mathbf{R}|=k} \prod_{i=1}^N \prod_{s \in R_i} \frac{(-1)^{i-1} e^{i \frac{N-2}{2} E_{i\emptyset}(s)}}{\prod_{j=1}^N (2i) \sin \frac{1}{2} E_{ij}(s)}. \quad (3.5.8)$$

As emphasized, our viewpoint is that the real topological string computation defines the rules to describe the properties of the orientifold surface operator in the $SU(N)$ gauge theory. Let us now discuss the effect of the additional signs from the perspective

of the gauge theory, to gain insight into the additional ingredients in \mathcal{O}_Ω beyond the ϵ_2 -shift. The orientifold is acting with different (alternating) signs on the different Young diagram degrees of freedom associated to the edges in the web diagram. This might imply an action with different signs on the states charged under the corresponding Cartans (alternating when ordered as determined by the 5d real Coulomb branch parameters). It would be interesting to gain a more direct gauge theory insight into the definition of this orientifold action.

Before concluding, we would like to mention an important point. We have used the ϵ_2 -shift exactly as in the conifold case in section 3.5.1, and obtained a result very close to the real topological vertex computation. However, one should keep in mind that the BPS states of the $SU(N)$ theory have a much richer structure. Therefore, the additional signs are of crucial importance to reproduce the correct results for the complete orientifold action on the theory.

For instance, if we isolate the unoriented contribution from eq. (3.5.8), the extra signs are crucial to produce certain non-zero real BPS multiplicities. This can be checked explicitly e.g. for $SU(2)$ using the results from section 3.B. For instance, consider the real BPS multiplicities n_{d_1, d_2}^g for M2-branes wrapped with degrees d_1, d_2 on the homology classes B and F , respectively. Already at $g = 0$ we have $n_{1,0}^0 = -2$ but $n_{0,1}^0 = 0$. The extra signs are crucial to produce a non-zero result for the unoriented contribution of the vector multiplet from the M2-brane on B (which using eq. (3.5.7) would give zero contribution). Similar considerations can be drawn for many others of the enumerative results in section 3.B.

3.6. Conclusions

In this chapter we have explored the extension of the correspondence between topological strings on toric CY three-folds and 4d/5d supersymmetric gauge theories with 8 supercharges to systems with orientifolds with real codimension 2 fixed locus. On the topological string side, we have focused on quotients which produce the real topological string of [91], because of its remarkably simple physical realization in M-theory. We have analyzed the properties of the systems, and emphasized their behavior under flops of the geometry.

The real topological string amplitudes define the properties of a new kind of surface defect in the corresponding gauge theory. We have rephrased the amplitudes in a form adapted to a gauge theory interpretation, by means of a newly defined twisted Nekrasov partition function, and we have taken the first steps towards providing an intrinsically gauge-theoretic interpretation of the twisting operator.

It would be interesting to complete the gauge theory interpretation of the twist operator. The partition function obtained by the simple ϵ_2 shift does not distinguish invariant states from non-invariant states. Therefore, M2-branes wrapping F and M2-branes wrapping B essentially give the same contribution to the partition function eq. (3.5.7). However, in general, they would give a different contribution in the real topological string amplitude since the former correspond to non-invariant states and the latter correspond

to invariant states. Hence, another implementation may be related to some operation that distinguishes the invariant states from the non-invariant states. There may be also a possibility to shift the Coulomb branch moduli like the ordinary orbifolded instantons [57].

A way to complete the gauge theory interpretation may be to give a more specific description of the orientifold in the ADHM quantum mechanics. In [57], instantons with a surface defect were identified with orbifolded instantons via a chain of dualities of string theory. It would be interesting to extend their reasoning to our case and determine an effect of the orientifold defect in the ADHM quantum mechanics. Once we identify the effect in the ADHM quantum mechanics, then we may proceed in the standard localization technique with it.

There are several other interesting directions worth exploring:

- It would be interesting to exploit the M-theory picture to develop a refined real topological vertex formalism, and to compare it with the gauge theory computations.
- As explained, there are different kinds of O4-planes in the physical type IIA picture, which correspond to different M-theory lifts, and different unoriented topological strings (albeit, with intricate relations).
- It would also be interesting to consider the addition of extra D4-branes, either on top of the O4-plane or possibly on other lagrangian 3-cycles, to describe the unoriented version of the relation of open topological strings and vortex counting on surface defects [19, 28].

3.A. Real topological string

3.A.1. M-theory interpretation of the real topological string

BPS state counting As already observed, we consider tadpole canceling configurations, such that the M-theory lift of the O4/D4 system is smooth, i.e. there are no fixed points. This guarantees that locally, before moving around the M-circle, the physics looks like in the oriented case. The $SU(2)_L \times SU(2)_R$ group is broken by the orientifold to its Cartan generators, which are enough to assign multiplicities to the 5d BPS states.

There will be two kinds of states, as follows. First, those not invariant under the orientifold, will have their orientifold image curve somewhere in the covering \mathbb{X} , and will contribute to closed oriented amplitudes (thus, we can neglect them). Second, those corresponding to curves mapped to themselves by the involution; their overall \mathbb{Z}_2 parity is determined by their $SU(2)_L$ and $SU(2)_R$ multiplet structure: as in the original GV case, the R part parametrizes our ignorance about cohomology of D-brane moduli space, while the L part is cohomology of Jacobian, and we know how to break it explicitly, due to our simple orientifold action. Let us split the second class of states, for fixed genus \hat{g} and homology class β , according to their overall parity

$$GV'_{\hat{g},\beta} = GV'^{+}_{\hat{g},\beta} + GV'^{-}_{\hat{g},\beta}. \quad (3.A.1)$$

Once we move on the circle, these invariant states acquire integer or half-integer momentum, according to their overall parity; this is taken into account in the 2d Schwinger computation, that finally yields, after removing even wrapping states that correspond again to closed oriented sector, the numbers

$$\widehat{\text{GV}}_{\hat{g},\beta} := \text{GV}_{\hat{g},\beta}^{\prime+} - \text{GV}_{\hat{g},\beta}^{\prime-} \quad (3.A.2)$$

that appear in eq. (3.3.3). The detailed computation is described in [79].

Tadpole cancellation The requirement that physical tadpoles are canceled has an interesting implication for real topological amplitudes, for geometries with $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$ and $H_2(\mathbb{X}, L; \mathbb{Z}) = \mathbb{Z}$; these include well-studied examples like the real quintic or real local \mathbb{CP}^2 . We discuss this for completeness, even though the geometries in the main text do not have this torsion homology on L .

Denote the degree of a map by $d \in H_2(\mathbb{X}, L; \mathbb{Z})$. By looking at the appropriate exact sequence in homology, one can see that crosscaps contribute an even factor to d , while boundaries may contribute even or odd factor. Moreover, by looking at the M-theory background form C_3 , one can see that it contributes to the central charge a factor of $i/2$ for every crosscap, i.e. \mathbb{RP}^2 , that surrounds the O4-plane. This translates into a minus sign once Poisson resummation is performed, more precisely a $(-1)^{mc}$ sign, where c is the crosscap number and m wrapping number around the circle (recall the states we are interested in have odd m .)

Finally, since boundaries do not receive such contributions and one can show, with a heuristic argument regarding real codimension one boundaries in the moduli space of stable maps, that there is a bijection between curves that agree except for a replacement of a crosscap with a (necessarily even-degree) boundary, we conclude that these two classes of curves cancel against each other. This implies that the only contributions may arise from odd-degree-boundary curves, and it is written as a restriction $\chi = d \bmod 2$ on the summation in eq. (3.3.3), as it has been proposed in [91] based on the fact that these two contributions to the topological amplitude are not mathematically well-defined separately, and the above mentioned prescription produces integer BPS multiplicities.

3.A.2. Real topological vertex

The real topological vertex [65] is a technique that allows to compute the all genus topological string partition functions, in the presence of toric orientifolds, namely a symmetry of the toric diagram with respect to which we quotient. This corresponds to an involution σ of \mathbb{X} , and it introduces boundaries and crosscaps in the topological string theory. We restrict to unrefined quantities, since at the moment real topological vertex technology is only available for that setup.

The recipe morally amounts to taking a square root of the topological vertex amplitude of the corresponding oriented parent theory, as follows. First, we observe that contributions from legs and vertices that are not fixed by the involution can be dealt with using standard vertex rules, and they automatically give rise to a perfect square

once paired with their image. We then only need to explain how to deal with a fixed edge connecting two vertices. Their contribution to the partition function is given by a factor

$$\sum_{R_i} C_{R_j R_k R_i} (-e^{-t_i})^{|R_i|} (-1)^{n|R_i|} q^{\frac{1}{2}n(\|R_i\|^2 - \|R_i^t\|^2)} C_{R'_j R'_k R_i^t}, \quad (3.A.3)$$

where $C := C(q, q)$ was introduced in eq. (3.2.8), and notation corresponds to fig. 3.9. Here $n := \det(v_{j'}, v_j)$, where v_m represents an outgoing vector associated to leg m .

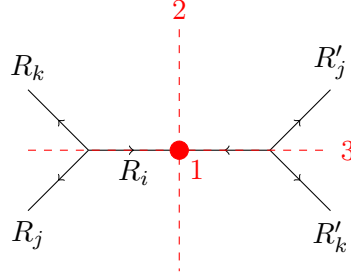


Figure 3.9: Three involutions for a generic internal leg; notice that each involution requires a specific symmetry to be present.

There are three cases: the involution can act as a point reflection at the center of the line (1), a reflection at the line perpendicular to the compact leg (2), or a reflection along the compact leg (3). The case interesting for us is (2), namely a leg that is not point-wise fixed by the involution, and where the representations in one vertex are mapped to representations in the other. In this case, no restriction is imposed on the internal representation R_i , while leg j is mapped to leg k' and similarly $k \rightarrow j'$. This imposes $R_j = R_k^{tt}$ and $R_k = R_j^{tt}$, where the transposition is implemented since the involution introduces an orientation-reversal of the plane in fig. 3.9. By exploiting the symmetry of function C

$$C_{AB^tC} = q^{\frac{\|A\|^2 - \|A^t\|^2 + \|B^t\|^2 - \|B\|^2 + \|C\|^2 - \|C^t\|^2}{2}} C_{BA^tC^t}, \quad (3.A.4)$$

we can rewrite eq. (3.A.3) as a perfect square, and take the square root:

$$\sum_{R_i} C_{R_j R_k R_i} e^{-\frac{1}{2}t_i |R_i|} (-1)^{\frac{1}{2}(n+1)s(R_i)} q^{\frac{1}{4}(n-1)(\|R_i\|^2 - \|R_i^t\|^2) + \frac{1}{4}(\|R_k^t\|^2 - \|R_k\|^2) + \frac{1}{4}(\|R_j\|^2 - \|R_j^t\|^2)}. \quad (3.A.5)$$

We introduced a sign in eq. (3.A.5)

$$(-1)^{\frac{1}{2}(n+1)s(R_i)}, \quad (3.A.6)$$

determined by $s(R) = |R| \pm c(R)$, where $c(R)$ is defined via $|R| - c(R) = 2 \sum_i R(2i)$. Finally, there is a global prescription for the choice of $c(R_i)$ vs. $c(R_i^t)$.

3.B. Enumerative checks

We compute the real GV invariants of the $SU(N)$ geometry with the involution considered in section 3.4.2. We describe some numerical checks that the topological vertex amplitudes indeed produce integer BPS multiplicities, corresponding to the proposed BPS state counting for the real topological string [79, 91]. The enumerative checks support the new result of the real topological string partition function for $SU(N)$ geometry in section 3.4.2.

After removing the purely closed oriented contribution, we perform an expansion

$$Z^{\text{unor}} = \frac{Z^{\text{real}}}{\sqrt{Z^{\text{or}}}} = 1 + Z_{1\text{-inst}}^{\text{real}} u^{\frac{1}{2}} + \left(Z_{2\text{-inst}}^{\text{real}} - \frac{1}{2} Z_{1\text{-inst}}^{\text{real}} \right) u + \mathcal{O}\left(u^{\frac{3}{2}}\right). \quad (3.B.1)$$

Note that the perturbative contribution does not contribute to the unoriented string part in the current choice of involution. From eq. (3.B.1), we can compute the real GV invariants: they are the numbers n appearing if we rewrite it using eq. (3.3.3),

$$Z^{\text{unor}} = \exp \sum_{d_1, d_2, g; \text{ odd } k} \frac{n_{d_1, d_2}^g}{k} \left(2i \sin \frac{k\epsilon}{2} \right)^{g-1} Q_B^{\frac{kd_1}{2}} Q_F^{\frac{kd_2}{2}}, \quad (3.B.2)$$

where we focus on the $SU(2)$ computation; from eq. (3.4.8) with $N = 2$ we obtain

$$Z_{1\text{-inst}}^{\text{real}} = \frac{1}{2 \sin \frac{\epsilon}{2} \sin t} \quad (3.B.3)$$

and

$$Z_{u^1}^{\text{unor}} = -\frac{3}{8} \frac{1}{\sin^2 \frac{\epsilon}{2} (\cos 2t - \cos \epsilon)} - \frac{1}{16} \frac{1}{\sin^2 t \sin^2 \frac{\epsilon}{2}}. \quad (3.B.4)$$

For illustration, we obtain real GV numbers up to $d_2 \leq 6$ and $g \leq 6$. We have $n_{1, d_2}^0 = -2$, for $d_2 = 0, 2, 4, 6$, and the others are zero. For n_{2, d_2}^g , we have

$d_2 \setminus g$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	0	3	0	0	0	0	0
3	0	0	0	0	0	0	0
4	0	12	0	3	0	0	0
5	0	0	0	0	0	0	0
6	0	30	0	18	0	0	0

(3.B.5)

For $d_1 = 3$ we have

$d_2 \setminus g$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	-6	0	-4	0	0	0	0
3	0	0	0	0	0	0	0
4	-28	0	-58	0	-28	0	-4
5	0	0	0	0	0	0	0
6	-82	0	-324	0	-362	0	-184

(3.B.6)

For $d_1 = 4$ we have

$d_2 \setminus g$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	0	12	0	5	0	0	0
3	0	0	0	0	0	0	0
4	0	153	0	268	0	177	0
5	0	0	0	0	0	0	0
6	0	900	0	3107	0	4670	0

(3.B.7)

For $d_1 = 5$ we have

$d_2 \setminus g$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	-12	0	-20	0	-6	0	0
3	0	0	0	0	0	0	0
4	-156	0	-744	0	-1212	0	-962
5	0	0	0	0	0	0	0
6	-990	0	-8518	0	-27704	0	-49814

(3.B.8)

For $d_1 = 6$ we have

$d_2 \setminus g$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	0	30	0	30	0	7	0
3	0	0	0	0	0	0	0
4	0	900	0	3293	0	5378	0
5	0	0	0	0	0	0	0
6	0	10255	0	70128	0	232826	0

(3.B.9)

Bound on genus One possible check we can perform is obtain the maximal g for given d [61]. To do this, let us take diagonal combinations $n_d^g := \sum_{d_1+d_2=d} n_{d_1,d_2}^g$:

$d \setminus g$	0	1	2	3
1	-2	0	0	0
2	0	0	0	0
3	-2	0	0	0
4	0	3	0	0
5	-8	0	-4	0
6	0	24	0	8

(3.B.10)

They satisfy tadpole cancellation $d = \chi = g - 1 \pmod{2}$. For fixed $d = d_1 + d_2$, a smooth curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ has genus $g = (d_1 - 1)(d_2 - 1)$, which is the top genus. For $d = 1$, we have a non-zero contribution from $d_1 = 1$ and $d_2 = 0$ for the involution we are considering. The top genus contribution appears from $g = 0$. The Gopakumar-Vafa invariant is then related to the Euler characteristic¹⁷ of the moduli space [61]

$$n_1^0 = n_{1,0}^0 = -e(\mathbb{CP}^1) = -2, \quad (3.B.11)$$

which is consistent with eq. (3.B.10).

The dimension of the moduli space of a curve inside $\mathbb{CP}^1 \times \mathbb{CP}^1$ may be understood as follows. The Gopakumar-Vafa invariants n_{d_1,d_2}^g are related to M2-branes wrapping a two-cycle

$$d_1 [\mathbb{CP}_B^1] + d_2 [\mathbb{CP}_F^1], \quad (3.B.12)$$

where $[\mathbb{CP}^1]$ represents a divisor class of \mathbb{CP}^1 . The degrees are also related to the degree of a polynomial that represents the curve by

$$\sum_{a_1,a_2,b_1,b_2} \alpha_{a_1,a_2,b_1,b_2} X_0^{a_1} X_1^{a_2} \tilde{X}_0^{b_1} \tilde{X}_1^{b_2} = 0, \quad a_1 + a_2 = d_2, \quad b_1 + b_2 = d_1, \quad (3.B.13)$$

where (X_0, X_1) are homogeneous coordinates of \mathbb{CP}_B^1 and $(\tilde{X}_0, \tilde{X}_1)$ are homogeneous coordinates of \mathbb{CP}_F^1 . We are now considering the case with $d_1 = 1$, $d_2 = 0$, which is

$$\alpha_{0,0,1,0} \tilde{X}_0 + \alpha_{0,0,0,1} \tilde{X}_1 = 0. \quad (3.B.14)$$

$\alpha_{0,0,1,0}$ and $\alpha_{0,0,0,1}$ still take value in \mathbb{C} and eq. (3.B.14) is still a complex equation. Therefore, the moduli space parametrized by $\alpha_{0,0,1,0}$ and $\alpha_{0,0,0,1}$ is \mathbb{CP}^1 . On the other hand, the deformation space of the curve class $d_2 [\mathbb{CP}_F^1]$ gives a real projective space due to the involution acting on \mathbb{CP}_B^1 . In general, the moduli space may be given by $\mathbb{CP}^{d_1} \times \mathbb{RP}^{d_2}$.

We can also consider the case $d = 2$. Then the top genus comes from $g = 0$. But this contribution will be absent since this does not satisfy the tadpole condition $d = g - 1 \pmod{2}$. This is also consistent with eq. (3.B.10).

¹⁷Recall that $e(\mathbb{CP}^m) = m + 1$ and $e(\mathbb{RP}^m) = 1, 0$ for m even/odd respectively.

The degree 3 case is also the same, namely

$$n_3^0 = n_{1,2}^0 = -e(\mathbb{CP}^1) e(\mathbb{RP}^2) = -2. \quad (3.B.15)$$

One may do similarly for $d = 4$. The maximal genus is 1 and hence 3 in eq. (3.B.10) will be related to the Euler characteristic of the moduli space: we find

$$n_4^1 = n_{2,2}^1 = (-1)^2 e(\mathbb{CP}^2) e(\mathbb{RP}^2) = 3. \quad (3.B.16)$$

For $d = 5$ we find

$$n_5^2 = n_{3,2}^2 = (-1)^3 e(\mathbb{CP}^3) e(\mathbb{RP}^2) = -4, \quad (3.B.17)$$

which is again consistent with the obtained result. For $d = 6$, the top genus is $g = 4$. But this does not satisfy the tadpole condition and hence $n_6^4 = 0$. That is also consistent with the result.

3.C. Definitions and useful identities

In this appendix we list some useful identities for quantities appearing in the topological vertex amplitudes. For non-trivial ones, we cite a reference where a proof can be found.

One can prove [11, eqs. (2.9) and (2.16)] the identity

$$\prod_{i,j=1}^{\infty} \left(1 - Qq^{i+j-1-R_1(j)-R_2(i)}\right) = \prod_{i,j=1}^{\infty} (1 - Qq^{i+j-1}) \prod_{s \in R_2} \left(1 - Qq^{-a_{R_1^t(s)}-1-l_{R_2}(s)}\right) \prod_{s \in R_1^t} \left(1 - Qq^{a_{R_2}(s)+1+l_{R_1^t}(s)}\right). \quad (3.C.1)$$

From definition $\|R\|^2 := \sum_i R(i)^2$, some simple combinatorial identities follow

$$\sum_i i\nu(i) = \frac{\|\nu^t\|^2}{2} + \frac{|\nu|}{2}, \quad \sum_{(i,j) \in \nu} \tilde{\nu}^t(j) = \sum_{(i,j) \in \tilde{\nu}} \nu^t(j), \quad \sum_{s \in \nu} l_\nu(s) = \frac{\|\nu\|^2}{2} - \frac{|\nu|}{2}, \quad (3.C.2)$$

which imply

$$\sum_{(m,n) \in R_i} a_{R_j}(m,n) = \frac{1}{2} (\|R_j^t\|^2 + |R_j| - \|R_i^t\|^2 - |R_i|) + \sum_{(m,n) \in R_j} a_{R_i}(m,n). \quad (3.C.3)$$

Using the above we get

$$\prod_{s \in R_i} \left(1 - e^{iE_{ij}(s)}\right)^{-1} \prod_{s \in R_j} \left(1 - e^{-iE_{ji}(s)}\right)^{-1} = (-1)^{|R_i|} e^{i\frac{t_{ji}}{2}(|R_i|+|R_j|)} e^{i\frac{\epsilon}{4}[\|R_j^t\|^2 - \|R_j\|^2 - \|R_i^t\|^2 + \|R_i\|^2]} \prod_{s \in R_i} \frac{1}{2i \sin \frac{E_{ij}(s)}{2}} \prod_{s \in R_j} \frac{1}{2i \sin \frac{E_{ji}(s)}{2}}. \quad (3.C.4)$$

Finally, the important identity for skew-Schur functions [70, page 93]

$$\sum_{\mu} s_{\mu/\eta}(x) s_{\mu/\xi}(y) = \prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1} \sum_{\mu} s_{\xi/\mu}(x) s_{\eta/\mu}(y) \quad (3.C.5)$$

gives upon iteration [94, Lemma 3.1]

$$\sum_{\substack{\nu^1, \dots, \nu^N \\ \eta^1, \dots, \eta^{N-1}}} \prod_{k=1}^N s_{\nu^k/\eta^{k-1}}(x^k) Q_k^{|\nu^k|} s_{\nu^k/\eta^k}(y^k) = \prod_{\substack{1 \leq k < l \leq N+1 \\ i, j \geq 1}} \left(1 - Q_k Q_{k+1} \cdots Q_{l-1} x_i^k y_j^{l-1}\right)^{-1}, \quad (3.C.6)$$

where $\eta^0 = \emptyset = \eta^N$ and $x^k = (x_1^k, x_2^k, \dots)$.

Chapter 4

Conclusion and outlook

4.1. Final remarks

In this thesis, we discussed the BPS integer expansion of the real topological string [91], using the M-theory lift of the O4-plane with one stuck D4-brane. Since the geometry is a \mathbb{Z}_2 quotient acting freely in the M-theory \mathbb{S}^1 , the 5d setup enjoys an enhancement to 8 supercharges and is identical to that in the closed oriented Gopakumar-Vafa system. The subtleties due to the orientifold quotient arise as a compactification effect modifying the KK momentum of the BPS states on the \mathbb{S}^1 according to their parity under the orientifold action. This allows for a clean derivation of the BPS integer expansion, without the extra assumptions that pop up in other unoriented A-models.

Although we recover the conjectured BPS expansion [91], our derivation shows the correct identification of the BPS invariants not as the equivariant sector of the parent Gopakumar-Vafa invariants, but rather a weighted version thereof.

The M-theory picture provides a complementary viewpoint on the sign choices implied by the tadpole cancellations in models where the fixed lagrangian 3-cycle L has $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$ [91]. More generally, the BPS integer expansion we propose is valid for other situations, providing a general definition of the real topological string.

The M-theory lift of other O4-planes suggests non-trivial relations between their BPS invariant expansions, for instance the addition of an open M2-brane sector (associated to two stuck M5-branes) to the lift of the O4⁻-plane should reproduce a sign flip in odd crosscap contributions. This seems to imply non-trivial relations among the unoriented and open BPS invariants in M-theory orientifolds with fixed points.

Note that when one tries to generalize the above constructions, due to infrared effects [25] it might not be correct to perform the Schwinger-type computations for cases where the Lagrangian has $b_1 > 0$, it is not topologically $\mathbb{R}^2 \times \mathbb{S}^1$ or \mathbb{RP}^3 , or $N > 1$ D4-branes wrap it.

We also explored the extension of the correspondence between topological strings on toric CY three-folds and 4d/5d supersymmetric gauge theories with 8 supercharges to systems with orientifolds with real codimension 2 fixed locus. On the topological string side, we focused on quotients which produce the real topological string [91], because of its remarkably simple physical realization in M-theory. We analyzed the properties of the systems, and emphasized their behavior under flops of the geometry.

The real topological string amplitudes define the properties of a new kind of surface defect in the corresponding gauge theory. We have rephrased the amplitudes in a form adapted to a gauge theory interpretation, by means of a newly defined twisted Nekrasov partition function, and we have taken the first steps towards providing an intrinsically gauge-theoretic interpretation of the twisting operator.

The ϵ_2 -shift that we performed on Nekrasov instanton partition function, although it passes many consistency checks and produces integer BPS multiplicities, does not completely reproduce the results from real topological vertex, due to different $(-1)^{|R|}$ signs in particular sub-sectors of the theory. One might therefore try to add extra ingredients, e.g. shift the Coulomb branch moduli as for ordinary orbifolded instantons [57], where

instantons with a surface defect were identified with orbifolded instantons via a chain of dualities. It would be interesting to extend their reasoning to our case and more generally determine the effect of the orientifold defect in the ADHM quantum mechanics. Once we identify the effect in the ADHM quantum mechanics, then we may use localization techniques.

A Spanish translation follows.

4.2. Observaciones finales

En esta tesis hemos discutido la expansión entera BPS de la cuerda topológica real [91], usando el *uplift* a teoría M del O4-plano con una D4-brana fija sobre éste. Dado que la geometría es un cociente \mathbb{Z}_2 que actúa libremente en el círculo \mathbb{S}^1 de teoría M, la configuración 5d sufre un incremento a 8 *supercargas* y es idéntica a la que hay en el sistema de Gopakumar y Vafa orientado cerrado. Las sutilezas debidas al cociente de orientifold surgen como un efecto de compactificación en la modificación del momento KK de los estados BPS en el \mathbb{S}^1 , de acuerdo con su paridad bajo la acción del orientifold. Esto permite una derivación limpia de la expansión entera BPS, sin los supuestos adicionales que aparecen en otros modelos de tipo A no orientados.

A pesar de que recuperamos la expansión BPS conjeturada [91], nuestra derivación muestra la correcta identificación de los invariantes de BPS no como el sector equivalente de los invariantes padres de Gopakumar y Vafa, sino más bien como una versión ponderada de los mismos.

La imagen en teoría M proporciona un punto de vista complementario sobre las opciones de signos implícitos en las cancelaciones de *tadpoles* en modelos en los que el 3-ciclo lagrangiano fijo L tiene $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$ [91]. Más generalmente, la expansión entera BPS que proponemos es válida para otras situaciones, proporcionando una definición general de la cuerda topológica real.

El *uplift* a teoría M de otros O4-planos sugiere relaciones no triviales entre sus expansiones invariantes BPS. Por ejemplo, la adición de un sector de M2-brana abierta (asociado a dos M5-branas fijas) al *uplift* del O4⁻-plano debería reproducir un cambio de signo en las contribuciones de *crosscap* impares. Esto parece implicar relaciones no triviales entre invariantes BPS abiertos no orientados en orientifolds de teoría M con puntos fijos.

Es importante darse cuenta de que cuando se trata de generalizar las construcciones anteriores, debido a los efectos infrarrojos [25] puede que no sea correcto utilizar cálculos de tipo Schwinger para los casos en los que el ciclo Lagrangiano tiene $b_1 > 0$, no es topológicamente $\mathbb{R}^2 \times \mathbb{S}^1$ o \mathbb{RP}^3 , o hay $N > 1$ D4-branas enrolladas.

También exploramos la extensión de la correspondencia entre cuerda topológica sobre CY tóricos y teorías *gauge* supersimétricas en 4d/5d con 8 *supercargas* a sistemas con orientifolds con locus fijo de codimensión real 2. En el lado de cuerda topológica, nos centramos en los cocientes que produce la cuerda topológica real [91], debido a su notablemente simple realización física en teoría M. Hemos analizado las propiedades de los

sistemas y hecho hincapié en su comportamiento bajo *flops* de la geometría.

Las amplitudes de cuerda topológica real definen las propiedades de un nuevo tipo de defecto de superficie en la teoría *gauge* correspondiente. Hemos reformulado las amplitudes en una forma adaptada a una interpretación en teoría *gauge*, por medio de una función de partición de Nekrasov *twistada* que hemos definido, y hemos dado los primeros pasos para proporcionar una interpretación intrínsecamente en teoría *gauge* del operador de *twist*.

El *shift* de ϵ_2 que realizamos en la función de partición instantónica de Nekrasov pasa muchas comprobaciones de coherencia y reproduce multiplicidades BPS enteras, pero no reproduce completamente los resultados del vértice topológico real, debido a diversos signos $(-1)^{|R|}$ en algunos sectores de la teoría. Por lo tanto, uno podría tratar de añadir ingredientes adicionales, por ejemplo desplazar los módulos del *Coulomb branch* como se hace para instantones orbifoldados ordinarios [57], donde se identifican los instantones con un defecto de superficie con instantones orbifoldados a través de una cadena de dualidades. Sería interesante ampliar su razonamiento a nuestro caso y más generalmente determinar el efecto del defecto de orientifold en la mecánica cuántica ADHM. Una vez identificado el efecto en la mecánica cuántica ADHM, se podrían utilizar técnicas de localización.

4.3. Possible developments

Some interesting questions that one may investigate further are as follows.

- The relation between O4-planes in type IIA and in topological string: as is well known, there are 4 different O4/D4 systems, whose classification is based on the O-plane charge and the presence of RR background fields; one could study the M-theory lift of the other O4/D4 systems, which are believed to yield different topological string constructions and thus different integrality properties for the enumerative invariants. In many of these cases the main difficulty lies in the fact that the M-theory lift is not smooth, so one should find a way to study the physics near these singularities and extract relevant information on BPS states.
- Adding N pairs of M5-branes away from the (M-theory lift of the) orientifold should lead to a simple extension of the model we considered, where one introduces extra open string sectors, and expects to see an enhancement of the background brane $SO(2N)$ symmetry to $SO(2N + 1)$.
- Study more general tadpole canceling configurations, as was already initiated by Krefl and Walcher [66], for example when the lagrangian has multiple connected components; one expects that topological tadpole cancellation corresponds to the physical tadpole cancellation, while on the topological string side it guarantees the decoupling of vector and hyper multiplets. It would be interesting to study the enumerative geometry of this configurations, and its implications for BPS state counting.

- One could ask how robust these type of invariants are against complex structure deformations: differently from the standard oriented case, where one finds the invariants are independent of cs deformations, in this setup it is not clear, as for example the notion of lagrangian subspace is changing.
- There is a subtlety in the choice of signs, related to how one can take the square root in the real vertex prescription, that needs further investigation; this subtlety is related to the orientifold action on the Coulomb branch parameters of the gauge theory. One could try to clarify this by deriving a localization formula in the orientifold background, by slightly changing the SO/Sp ADHM construction, and check whether the real vertex prescriptions are exactly matched.
- Moreover, one could consider a different type of involution of the toric diagram, where the number of Coulomb branch parameters gets reduced, and try to find the gauge theory dual.
- In the framework of BPS state counting in presence of defects, the above result suggests a relation between unoriented GV invariants and codimension two surface defects [28] in 4d gauge theory, possibly clarifying the interpretation of the O4-plane as a defect, in a similar fashion as open BPS invariants and surface defects are related [28, 31]. The peculiarity here is that the holonomy is given by an outer automorphism of the original gauge group, so one has to consider a different kind of defect than the ones studied by Gukov and collaborators [28].

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